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Symmetry Transformations with Noncommutative and Nonassociative Parameters

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We study generalized symmetry transformations which involve nonassociative and noncommutative parameters. The structure underlying the group gradings is determined and examples are given. Graded algebras beyond Grassmann algebras are also presented. Nontrivial examples relevant for graded extensions beyond supersymmetry are given which resemble several features of quarks and might lead to a connection between the external and internal symmetries of the phenomenological models. Lie groups of transformations involving nonassociative and noncommutative parameters are obtained together with their corresponding graded Lie algebraic structures.

1. INTRODUCTION

The aim of this paper is to provide a structure that allows us to extend the concept of continuous symmetries.

The no-go theorems of Coleman and Mandula (1967) and of Haag *et al.* (1975) have established the more general symmetry of the S-matrix in a quantum field theory model using respectively a group and a supergroup structure. The grading structure underlying the Poincaré algebra and its possible graded extensions have been determined in Wills-Toro (1995). These results open the possibility of gradings beyond supersymmetry whose parameters obey generalized commutation relations (q-commutativity) and are associative (Wills-Toro, 1994a, b). There, the each-other commuting space-time parameters can have unexpected generalized commutation relations with further generalized Grassmann parameters. This novel nontrivial behavior of the space-time parameters was not envisaged when studying the most general form of the quantization relations between quantum fields (Klein, 1938;

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Lüders, 1958; Kinoshita, 1958) and the corresponding connection between spin and statistics (Pauli, 1940; Araki 1961). The so-called Klein transformations (Klein, 1938; Araki, 1961) and further equivalences among graded Lie algebras should be further developed (Rittenberg and Wyler, 1978; Scheunert, 1979, 1983a, b), in order to determine to which extent such extensions are truly inequivalent to the supersymmetric ones.

The novel gradings open an unexpected possibility that we introduce in this paper: Symmetry transformations whose parameters are *noncommutative and nonassociative*. We will show that this possibility is actually not present for supergradings. We also show that these structures can be consistently realized for parameter algebras using the gradings suitable for extending the Poincaré algebra. These graded parameters resemble several features of the quark phenomenology and provides an alternative line of development to previous studies relating para-statistic and global gauge groups (Ohnuki and Kamefuchi, 1968, 1969, Drühl *et al.*, 1970). In the graded Lie algebraic counterpart, the novel extensions might offer further layers of generators that can produce, for instance, susy generators through the iterated Lie products of three generators, in analogous fashion as we produce translations through the Lie product of two susy generators.

2. SYMMETRY GROUPS AND GENERATOR ALGEBRAS

The most fundamental description of nature seems to be related to the characterization of its building blocks in terms of its (exact, approximated, broken, global, local, internal, and external) symmetries. The mathematical structure underlying the set and composition of symmetry transformations is the concept of group of transformations. Regarding the countability of its elements, we recognize two main classes: the discrete and continuous group of transformations. Every continuous group of transformations turns out to be isomorphic to the product of a discrete and a Lie group. The *Lie group* is the connected component which contains the identity transformation. Every element g_α of the Lie group can be realized as the exponential of a linear combination of linearly independent generators G_j, $j = 1, \ldots, N$, through commutative parameters (numbers) α^j; $j = 1, \ldots, N$:

$$g_\alpha = g(\alpha^1, \ldots, \alpha^N) = \exp\{i\alpha^j G_j\} \tag{2.1}$$

The generators build a *Lie algebra* $\mathcal{A} = gen\{G_1, \ldots, G_N\}$ with a Lie product $[\cdot,\cdot]$ fulfilling for every $X, Y, Z \in \mathcal{A}$: *closure* $[X, Y] \in \mathcal{A}$, *linearity* $[X + Y, Z] = [X, Z] + [Y, Z]$, *antisymmetry* $[X, Y] = -[Y, X]$, and *Jacobi associativity* (derivative rule) $[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]]$.

Supergroups offered a further extension of the concept of symmetry transformations in which *Grassmann* or *fermionic* parameters $\theta^1, \ldots, \theta^\mu$ are introduced besides the *commutative* or *bosonic* parameters $\alpha^1, \ldots, \alpha^N$:

$$g_{\alpha,\theta} = \exp\{i\alpha^j G_j + i\theta^\beta Q_\beta\} \tag{2.2}$$

The generators Q_β possess fermionic character and the monomials $\theta^\beta Q_\beta$ (no summation) and $\alpha^j G_j$ (no summation) span a Lie algebra. Accordingly, the parameter θ^β "compensates" the fermionic character of Q_β. We are looking for symmetry transformations of the form

$$g_\beta = \exp\{i\beta^{(j)}{}_{\underline{\tilde{a}}} Q^{(j)}_{\tilde{a}} \tag{2.3}$$

which involve parameters $\beta^{(j)}{}_{\underline{\tilde{a}}}$ with generalized commutative and associative behavior, and generators $Q^{(j)}_{\tilde{a}}$ which constitute a generalized Lie algebraic structure. The monomials $\beta^{(j)}{}_{\underline{\tilde{a}}} Q^{(j)}_{\tilde{a}}$ (no summation) span a Lie algebra.

In theoretical physics, several unwritten conventions fix the behavior of the parameters involved. For instance, the addition of a fermionic with a bosonic parameter is not allowed. This rule is motivated by the spin–statistics theorem and covariance requirements (integer-spin representations are used for bosonic operators—or fields—and half-odd representations are used for fermionic operators). The addition of a fermionic with a bosonic parameter or operator would lead to noncovariant expressions. We will consider now structures in which the addition is restricted to objects with identical commutative and associative behavior since such a constraint seems to be well motivated in models for mathematical physics. This constraint corresponds to the adoption of some *superselection rules* from the very start, but this leads to wider structures for the sets of parameters and generator algebras, although it might appear paradoxical.

3. I-GRADED, *q*-COMMUTATIVE, AND *r*-ASSOCIATIVE PARAMETERS

We will address now a generalization of the concept of number fields in which the elements are called parameters. The generalization of the real field that leads to the *complex, quaternion, octonion*, and *Cayley numbers* abandons sequentially the *linear order field*, then *commutativity*, then *associativity*, and then the *division-ring property*. This construction maintains the addition between any two numbers. This line of generalization disregards the mentioned fact: In several applications, the addition can be meaningfully restricted to objects with analogous commutative and associative behavior. We will consider a particular generalization of the concept of Grassmann parameters in which *each parameters has a tilded index that determines its* *behavior under addition, product, permutation of factors, and alteration of parenthesis.* Each parameter, say β, has associated one tilded index, say $\tilde{a}$. In order to recognize immediately the index associated to the parameter β, we can write explicitly $\beta_{\tilde{a}}$. In several physical applications the usage of certain letters or alphabet is reserved to certain parameters, but this procedure is rather unsustainable once there are more than three types of parameters.

We consider parameters $\beta_{\tilde{a}}, \beta'_{\tilde{e}}, \beta''_{\tilde{c}}, \ldots$ whose products are I-*graded, q-commutative*, and *r-associative:*

$$(\beta_{\tilde{a}}\beta'_{\tilde{e}}) \quad \text{has index} \quad \tilde{a} + \tilde{e} \tag{3.1}$$

$$(\beta_{\tilde{a}}\beta'_{\tilde{e}}) = \quad q_{\tilde{a},\tilde{e}}(\beta'_{\tilde{e}}\beta_{\tilde{a}}), \qquad \text{where} \quad q_{\tilde{a},\tilde{e}} \in \mathrm{K}\backslash\{0\} \tag{3.2}$$

$$(\beta_{\tilde{a}}(\beta'_{\tilde{e}}\beta''_{\tilde{c}})) = \quad r_{\tilde{a},\tilde{e},\tilde{c}}((\beta_{\tilde{a}}\beta'_{\tilde{e}})\beta''_{\tilde{c}}), \qquad \text{where} \quad r_{\tilde{a},\tilde{e},\tilde{c}} \in \mathrm{K}\backslash\{0\} \tag{3.3}$$

The set K is a given numeric field (R, C, ...) and + is an internal operation in the set of indices I. We shall avoid the outermost parenthesis in a parameter monomial.

There are, of course, wider generalizations in which, for instance, the interchange of the position of factors or the alteration of parenthesis introduces a linear combination of products of parameters. Such cases are beyond the scope of this study. We determine now which could be the properties of the parameters, of the index set I, of the numerical field K, and of the q- and r-factors that might lead to a nontrivial realization of the set of parameters. We assume first that the elements of K\{0} do not contribute to the tilded index of a monomial, i.e.,

$$(y\beta_{\tilde{a}} \text{ has tilded index } \tilde{a}) \qquad \text{for all} \quad y \in \mathrm{K}\backslash\{0\} \tag{3.4}$$

Accordingly, we can consider that every element of K\{0} carries an additive neutral index $\tilde{o} \in \mathrm{I}$:

$$(y \text{ has index } \tilde{o}) \qquad \text{for all} \quad y \in \mathrm{K}\backslash\{0\} \tag{3.5}$$

$$(\tilde{a} + \tilde{o} = \tilde{o} + \tilde{a} = \tilde{a}) \qquad \text{for all} \quad \tilde{a} \in \mathrm{I} \tag{3.6}$$

Since the numerical factors in K\{0} do not contribute to the q- and r-factors, we adopt

$$(q_{\tilde{a},\tilde{o}} = q_{\tilde{o},\tilde{a}} = 1) \qquad \text{for all} \quad \tilde{a} \in \mathrm{I} \tag{3.7}$$

$$(r_{\tilde{a},\tilde{e},\tilde{o}} = r_{\tilde{a},\tilde{o},\tilde{e}} = r_{\tilde{o},\tilde{a},\tilde{e}} = 1) \qquad \text{for all} \quad \tilde{a}, \tilde{e} \in \mathrm{I} \tag{3.8}$$

$$(1\beta_{\tilde{a}} = \beta_{\tilde{a}}1 = \beta_{\tilde{a}}) \tag{3.9}$$

From the properties (3.1) and (3.2) we conclude that $(\beta_{\tilde{a}}\beta'_{\tilde{e}}) = q_{\tilde{a},\tilde{e}}(\beta'_{\tilde{e}}\beta_{\tilde{a}})$ has index $\tilde{o} + (\tilde{e} + \tilde{a}) = (\tilde{e} + \tilde{a})$. Hence, $\tilde{a} + \tilde{e}$ and $\tilde{e} + \tilde{a}$ are identical. Analogously, from property (3.3) it follows that the addition of the indices is associative. So we can write

$$(\tilde{a} + \tilde{e} = \tilde{e} + \tilde{a}) \qquad \text{for all} \quad \tilde{a}, \tilde{e} \in \mathrm{I} \tag{3.10}$$

$$(\tilde{a} + (\tilde{e} + \tilde{c}) = (\tilde{a} + \tilde{e}) + \tilde{c}) \qquad \text{for all} \quad \tilde{a}, \tilde{e}, \tilde{c} \in \mathrm{I} \tag{3.11}$$

In order to obtain compounds parameter-by-generator of the form $\beta_{-a}Q_{\tilde{a}}$ which produce trivial q- and r-factors (since they constitute a Lie algebra), we require the existence of additive inverses in I:

$$\text{for all} \quad \tilde{a} \in \mathrm{I} \quad \exists - \tilde{a} \in \mathrm{I} \quad (\tilde{a} + -\tilde{a} = \tilde{o}) \tag{3.12}$$

From the properties (3.1), (3.6), (3.10), (3.11), and (3.12) we conclude that the set I with its addition operation + builds an abelian group structure. From the properties (3.2) and (3.3) we conclude that the left and right multiplicative inverses of the q- and r-factors are identical. We can therefore assume

$$\text{K is a commutative field} \tag{3.13}$$

From the reiterated usage of the property (3.2), we conclude

$$(q_{\tilde{e},\tilde{a}} = (q_{\tilde{a},\tilde{e}})^{-1} \text{ or } \beta_{\tilde{a}}\beta'_{\tilde{e}} = 0) \qquad \text{for all} \quad \tilde{a}, \tilde{e} \in \mathrm{I} \tag{3.14}$$

From the property (3.2) we conclude as well

$$(q_{\tilde{a},\tilde{a}} = 1 \text{ or } \beta_{\tilde{a}}\beta_{\tilde{a}} = 0) \qquad \text{for all} \quad \tilde{a} \in \mathrm{I} \tag{3.15}$$

The latter result is the cornerstone of the definition of Grassmann parameters. Instead of requiring $q_{\tilde{a},\tilde{a}} = 1$, will be considered the existence of nilpotent parameters.

At the level of cubic monomials of parameters, we obtain different equations, depending on the way we use the r-associativity and q-commutativity:

$$\beta_{\tilde{a}}(\beta'_{\tilde{e}}\beta''_{\tilde{c}}) = q_{\tilde{a},\tilde{e}+\tilde{c}}(\beta'_{\tilde{e}}\beta''_{\tilde{c}})\beta_{\tilde{a}} \tag{3.16}$$

$$\beta_{\tilde{a}}(\beta'_{\tilde{e}}\beta''_{\tilde{c}}) = r_{\tilde{a},\tilde{e},\tilde{c}}q_{\tilde{a},\tilde{e}}r^{-1}_{\tilde{e},\tilde{a},\tilde{c}}q_{\tilde{a},\tilde{c}}r_{\tilde{e},\tilde{c},\tilde{a}}(\beta'_{\tilde{e}}\beta''_{\tilde{c}})\beta_{\tilde{a}} \tag{3.17}$$

$$\beta_{\tilde{a}}(\beta'_{\tilde{e}}\beta''_{\tilde{c}}) = r_{\tilde{a},\tilde{e},\tilde{c}}q_{\tilde{a}+\tilde{e},\tilde{c}}q_{\tilde{a},\tilde{e}}r_{\tilde{c},\tilde{e},\tilde{a}}q_{\tilde{c},\tilde{e}}(\beta'_{\tilde{e}}\beta''_{\tilde{c}})\beta_{\tilde{a}} \tag{3.18}$$

These equations provide a consistency requirement for cubic monomials:

$$(q_{\tilde{a},\tilde{e}+\tilde{c}} = r_{\tilde{a},\tilde{e},\tilde{c}}q_{\tilde{a},\tilde{e}}r^{-1}_{\tilde{e},\tilde{a},\tilde{c}}q_{\tilde{a},\tilde{c}}r_{\tilde{e},\tilde{c},\tilde{a}} = r_{\tilde{a},\tilde{e},\tilde{c}}q_{\tilde{a}+\tilde{e},\tilde{c}}q_{\tilde{a},\tilde{e}}r_{\tilde{c},\tilde{e},\tilde{a}}q_{\tilde{c},\tilde{e}}$$

$$\text{or } \beta_{\tilde{a}}(\beta'_{\tilde{e}}\beta''_{\tilde{c}}) = 0) \qquad \text{for all} \quad \tilde{a}, \tilde{e}, \tilde{c} \in \mathrm{I} \tag{3.19}$$

The r-associativity in quartic monomials leads to a further set of equations:

$$(\beta_{\tilde{a}}\beta'_{\tilde{e}})(\beta''_{\tilde{c}}\beta'''_{\tilde{u}}) = r_{\tilde{a}+\tilde{e},\tilde{c},\tilde{u}}((\beta_{\tilde{a}}\beta'_{\tilde{e}})\beta''_{\tilde{c}})\beta'''_{\tilde{u}} \tag{3.20}$$

$$(\beta_{\tilde{a}}\beta'_{\tilde{e}})(\beta''_{\tilde{c}}\beta'''_{\tilde{u}}) = r^{-1}_{\tilde{a},\tilde{e},\tilde{c}+\tilde{u}} r_{\tilde{e},\tilde{c},\tilde{u}} r_{\tilde{a},\tilde{e}+\tilde{c},\tilde{u}} r_{\tilde{a},\tilde{e},\tilde{c}}((\beta_{\tilde{a}}\beta'_{\tilde{e}})\beta''_{\tilde{c}})\beta'''_{\tilde{u}} \tag{3.21}$$

These equations provide a consistency condition for quartic monomials:

$$(r^{-1}_{\tilde{a}+\tilde{e},\tilde{c},\tilde{u}} r^{-1}_{\tilde{a},\tilde{e},\tilde{c}+\tilde{u}} r_{\tilde{a},\tilde{e}+\tilde{c},\tilde{u}} r_{\tilde{e},\tilde{c},\tilde{u}} r_{\tilde{a},\tilde{e},\tilde{c}} = 1$$

$$\text{or } (\beta_{\tilde{a}}\beta'_{\tilde{e}})(\beta''_{\tilde{c}}\beta'''_{\tilde{u}}) = 0) \qquad \text{for all} \quad \tilde{a}, \tilde{e}, \tilde{c}, \tilde{u} \in \mathbb{I} \tag{3.22}$$

A further condition for quartic monomials can be obtained from the equations

$$(\beta_{-\tilde{a}}\eta_{\tilde{a}})(\beta'_{-\tilde{e}}\eta'_{\tilde{e}}) = r_{-\tilde{e},-\tilde{a},\tilde{a}} r^{-1}_{-\tilde{e}-\tilde{a},\tilde{a},\tilde{e}}(\beta'_{-\tilde{e}}\beta_{-\tilde{a}})(\eta_{\tilde{a}}\eta'_{\tilde{e}})$$

$$= (\beta'_{-\tilde{e}}\eta'_{\tilde{e}})(\beta_{-\tilde{a}}\eta_{\tilde{a}}) = r_{-\tilde{a},-\tilde{e},\tilde{e}} r^{-1}_{-\tilde{a}-\tilde{e},\tilde{e},\tilde{a}} q_{-\tilde{a},-\tilde{e}} q_{\tilde{e},\tilde{a}}(\beta'_{-\tilde{e}}\beta_{-\tilde{a}})(\eta_{\tilde{a}}\eta'_{\tilde{e}}) \tag{3.23}$$

These equations provide a consistency condition for quartic monomials:

$$(r^{-1}_{-\tilde{e},-\tilde{a},\tilde{a}} r_{-\tilde{e}-\tilde{a},\tilde{a},\tilde{e}} r_{-\tilde{a},-\tilde{e},\tilde{e}} r^{-1}_{-\tilde{a}-\tilde{e},\tilde{e},\tilde{a}} q_{-\tilde{a},-\tilde{e}} q_{\tilde{e},\tilde{a}} = 1$$

$$\text{or } (\beta_{-\tilde{a}}\eta_{\tilde{a}})(\beta'_{-\tilde{e}}\eta'_{\tilde{e}}) = 0) \qquad \text{for all} \quad \tilde{a}, \tilde{e} \in \mathrm{I} \tag{3.24}$$

We obtain, finally, some equations for monomials of order six:

$$\begin{aligned}
&(\beta_{-\tilde{a}}\eta_{\tilde{a}})((\beta'_{-\tilde{e}}\eta'_{\tilde{e}})(\beta''_{-\tilde{c}}\eta''_{\tilde{c}}))\\
&\quad = r_{-\tilde{c},-\tilde{e},\tilde{e}} r^{-1}_{-\tilde{c}-\tilde{e},\tilde{e},\tilde{c}} r_{-\tilde{c}-\tilde{e},-\tilde{a},\tilde{a}} r^{-1}_{-\tilde{a}-\tilde{e}-\tilde{c},\tilde{a},\tilde{e}+\tilde{c}}((\beta''_{-\tilde{c}}\beta'_{-\tilde{e}})\beta_{-\tilde{a}})(\eta_{\tilde{a}}(\eta'_{\tilde{e}}\eta''_{\tilde{c}}))\\
&\quad = ((\beta_{-\tilde{a}}\eta_{\tilde{a}})(\beta'_{-\tilde{e}}\eta'_{\tilde{e}}))(\beta''_{-\tilde{c}}\eta''_{\tilde{c}})\\
&\quad = r_{-\tilde{e},-\tilde{a},\tilde{a}} r^{-1}_{-\tilde{e}-\tilde{a},\tilde{a},\tilde{e}} r_{-\tilde{c},-\tilde{e}-\tilde{a},\tilde{e}+\tilde{a}} r^{-1}_{-\tilde{a}-\tilde{e}-\tilde{c},\tilde{e}+\tilde{a},\tilde{c}} r_{-\tilde{c},-\tilde{e},-\tilde{a}} r^{-1}_{\tilde{a},\tilde{e},\tilde{c}}\\
&\quad \times ((\beta''_{-\tilde{c}}\beta'_{-\tilde{e}})\beta_{-\tilde{a}})(\eta_{\tilde{a}}(\eta'_{\tilde{e}}\eta''_{\tilde{c}}))\\
&\quad = (\beta'_{-\tilde{e}}\eta'_{\tilde{e}})((\beta_{-\tilde{a}}\eta_{\tilde{a}})(\beta''_{-\tilde{c}}\eta''_{\tilde{c}}))\\
&\quad = r_{-\tilde{c},-\tilde{a},\tilde{a}} r^{-1}_{-\tilde{c}-\tilde{a},\tilde{a},\tilde{c}} r_{-\tilde{c}-\tilde{a},-\tilde{e},\tilde{e}} r^{-1}_{-\tilde{a}-\tilde{e}-\tilde{c},\tilde{e},\tilde{a}+\tilde{c}} r^{-1}_{-\tilde{c},-\tilde{a},-\tilde{e}} r_{-\tilde{c},-\tilde{e},-\tilde{a}} r_{\tilde{e},\tilde{a},\tilde{c}} r^{-1}_{\tilde{a},\tilde{e},\tilde{c}}\\
&\quad \times q_{-\tilde{a},-\tilde{e}} q_{\tilde{e},\tilde{a}}((\beta''_{-\tilde{c}}\beta'_{-\tilde{e})\beta_{-\tilde{a}}})(\eta_{\tilde{a}}(\eta'_{\tilde{e}}\eta''_{\tilde{c}}))
\end{aligned} \tag{3.25}$$

These equations provide the consistency requirement

$$\begin{aligned}
&(r^{-1}_{-\tilde{c},-\tilde{e},\tilde{e}} r_{-\tilde{e}-\tilde{c},\tilde{e},\tilde{c}} r^{-1}_{-\tilde{e}-\tilde{c},-\tilde{a},\tilde{a}} r_{-\tilde{e}-\tilde{c}-\tilde{a},\tilde{a},\tilde{e}+\tilde{c}} r_{-\tilde{e},-\tilde{a},\tilde{a}}\\
&\quad \times r^{-1}_{-\tilde{e}-\tilde{a},\tilde{a},\tilde{e}} r_{-\tilde{c},-\tilde{e}-\tilde{a},\tilde{e}+\tilde{a}} r^{-1}_{-\tilde{c}-\tilde{e}-\tilde{a},\tilde{a}+\tilde{e},\tilde{c}} r_{-\tilde{c},-\tilde{e},-\tilde{a}} r^{-1}_{\tilde{a},\tilde{e},\tilde{c}}\\
&\quad = r^{-1}_{-\tilde{c},-\tilde{e},\tilde{e}} r_{-\tilde{e}-\tilde{c},\tilde{e},\tilde{c}} r^{-1}_{-\tilde{e}-\tilde{c},-\tilde{a},\tilde{a}} r_{-\tilde{e}-\tilde{c}-\tilde{a},\tilde{a},\tilde{e}+\tilde{c}}\\
&\quad \times r_{-\tilde{c},-\tilde{a},\tilde{a}} r^{-1}_{-\tilde{a}-\tilde{c},\tilde{a},\tilde{c}} r_{-\tilde{a}-\tilde{c},-\tilde{e},\tilde{e}}\\
&\quad \times r^{-1}_{-\tilde{a}-\tilde{c}-\tilde{e},\tilde{e},\tilde{a}+\tilde{c}} r^{-1}_{-\tilde{c},-\tilde{a},-\tilde{e}} r_{-\tilde{c},-\tilde{e},-\tilde{a}} r_{\tilde{e},\tilde{a},\tilde{c}} r^{-1}_{\tilde{a},\tilde{e},\tilde{c}} q_{-\tilde{a},-\tilde{e}} q_{\tilde{e},\tilde{a}} = 1
\end{aligned}$$

$$\text{or } (\beta_{-\tilde{a}}\eta_{\tilde{a}})((\beta'_{-\tilde{e}}\eta'_{\tilde{e}})(\beta''_{-\tilde{c}}\eta''_{\tilde{c}})) = 0) \qquad \text{for all} \quad \tilde{a}, \tilde{e}, \tilde{c} \in \mathbb{I} \tag{3.26}$$

We observe that for each monomial order there exists a set of independent constraints whose classification is very involved. Furthermore, the determination of the independent constraints up to a given monomial order is a highly nontrivial question (Groebner basis problem).

Finally, we construct a graded Lie algebraic structure such that the compounds $\beta_{-\tilde{a}}Q_{\tilde{a}}$ span a Lie algebra. We require thus closure, linearity, antisymmetry, and Jacobi associativity among them. We assume first a commutator of the form

$$[(\beta_{-\tilde{a}}Q_{\tilde{a}}), (\beta'_{-\tilde{e}}Q'_{\tilde{e}})] = (\beta_{-\tilde{a}}Q_{\tilde{a}})(\beta'_{-\tilde{e}}Q'_{\tilde{e}}) - (\beta'_{-\tilde{e}}Q'_{\tilde{e}})(\beta_{-\tilde{a}}Q_{\tilde{a}}) \quad (3.27)$$

In order to obtain a closed structured among the generators $Q_{\tilde{a}}, Q'_{\tilde{e}}, \ldots,$ we will find that it is sufficient to impose the conditions on the q- and r-factors stated in the requirements (3.24) and (3.26).

The resulting algebraic structure $\mathbb{L}$ built up by the generators $Q, Q', \ldots$ is called an $(\mathbb{I}; q, r)$-graded Lie algebra over $\mathbb{K}$. The addition in $\mathbb{L}$ is only defined between elements with identical index, and the product $[\![\cdot, \cdot]\!]$ in $\mathbb{L}$ fulfills for all $X_{\tilde{a}}, Y_{\tilde{e}}, Z_{\tilde{c}}, X'_{\tilde{a}} \in \mathbb{L}$: *Closure and* $\mathbb{I}$-*grading:* $\exists\, U_{\tilde{a}+\tilde{e}} \in \mathbb{L}$ ($[\![X_{\tilde{a}}, Y_{\tilde{e}}]\!] = U_{\tilde{a}+\tilde{e}}$); *Linearity:* $[\![X_{\tilde{a}}, + X'_{\tilde{a}}, Y_{\tilde{e}}]\!] = [\![X_{\tilde{a}}, Y_{\tilde{e}}]\!] + [\![X'_{\tilde{a}}, Y_{\tilde{e}},]\!]$; *q-antisymmetry*: $[\![X_{\tilde{a}}, Y_{\tilde{e}}]\!] = -q_{\tilde{a},\tilde{e}}[\![Y_{\tilde{e}}, X_{\tilde{a}}]\!]$; (q, r)-Jacobi *associativity:*

$$[\![X_{\tilde{a}}, [\![Y_{\tilde{e}}, Z_{\tilde{c}}]\!]]\!] = r_{\tilde{a},\tilde{e},\tilde{c}}\, [\![[\![X_{\tilde{a}}, Y_{\tilde{e}}]\!], Z_{\tilde{c}}]\!] + r_{\tilde{a},\tilde{e},\tilde{c}}q_{\tilde{a},\tilde{e}}r^{-1}_{\tilde{e},\tilde{a},\tilde{c}}[\![Y_{\tilde{e}}, [\![X_{\tilde{a}}, Z_{\tilde{c}}]\!]]\!]$$

We observe that in consonance with our comments in Section 1, the addition in $\mathbb{L}$ *is only defined* among elements carrying identical tilded indices.

There is a further set of requirements (to be explained below) in order to complete sufficient conditions to demonstrate consistency of the monomial of parameters to all orders: there should exist an application R such that (for appropriate and fixed choices of the phases in the roots):

$$q^{\text{“1/2”}}_{\tilde{a},\tilde{e}}\, R_{\tilde{a},\tilde{e}} = q_{\tilde{a},\tilde{e}}q^{\text{“1/2”}}_{\tilde{e},\tilde{a}}\, R_{\tilde{e},\tilde{a}} \quad (3.28)$$

$$q^{\text{“1/2”}}_{\tilde{a},\tilde{e}+\tilde{c}}R_{\tilde{a},\tilde{e}+\tilde{c}}q^{\text{“1/2”}}_{\tilde{e},\tilde{c}}\, R_{\tilde{e},\tilde{c}} = r_{\tilde{a},\tilde{e},\tilde{c}}q^{\text{“1/2”}}_{\tilde{a},\tilde{e}}\, R_{\tilde{a},\tilde{e}}q^{\text{“1/2”}}_{\tilde{a}+\tilde{e},\tilde{c}}R_{\tilde{a}+\tilde{e},\tilde{c}} \quad (3.29)$$

We consider finally the basic requirements for the existence of an involution operation for the parameters. This allows for the definition of unitary group transformations of the form (2.3). The involution plays a central role for supersymmetric extensions since it leads to the existence of an energy ground state. The involution operation $\overline{(\cdot)}$ for parameters fulfills

$$\overline{(\beta_{\tilde{a}})} = \overline{\beta}_{\tilde{a}^\star}, \qquad \overline{(\overline{\beta}_{\tilde{a}^\star})} = \beta_{\tilde{a}} \quad (3.30)$$

$$\overline{(k\beta_{\tilde{a}}\beta'_{\tilde{e}})} = k^*\overline{(\beta'_{\tilde{e}})}\;\overline{(\beta_{\tilde{a}})} = k^*\overline{\beta}'_{\tilde{e}^\star}\overline{\beta}_{\tilde{a}^\star} \quad (3.31)$$

Hence, there should exist involutions $(\cdot)^\bigstar$ and $(\cdot)^*$ in $\mathbb{I}$ and $\mathbb{K}$, respectively, such that

$$((\bar{a})^\bigstar = \bar{a}^\bigstar,\ (\bar{a}^\bigstar)^\bigstar = \bar{a}) \qquad \text{for all} \quad \tilde{a} \in \mathbb{I} \tag{3.32}$$

$$((k)^* = k^*,\ (k^*)^* = k) \qquad \text{for all} \quad k \in \mathbb{K} \tag{3.33}$$

The consistency between the involutions $\overline{(\cdot)}$, $(\cdot)^\bigstar$ and $(\cdot)^*$, and the q-commutativity and r-associativity leads to the requirements

$$(\text{whether } q_{\bar{a}^\bigstar, \bar{e}^\bigstar} = (q^*_{\bar{a},\bar{e}})^{-1} \text{ or } \beta_{\bar{a}}\beta'_{\bar{e}} = 0) \qquad \forall \tilde{a}, \tilde{e} \in \mathbb{I} \tag{3.34}$$

$$(\text{whether } r_{\bar{c}^\bigstar, \bar{e}^\bigstar, \bar{a}^\bigstar} = (r^*_{\bar{a},\bar{e},\bar{c}})^{-1} \text{ or } \beta_{\bar{a}}(\beta'_{\bar{e}}\beta''_{\bar{c}}) = 0) \qquad \forall \tilde{a}, \tilde{e}, \tilde{c} \in \mathbb{I} \tag{3.35}$$

4. ABELIAN GROUP GRADINGS OVER A COMMUTATIVE FIELD

We have discussed in the previous section a series of requirements that lead to generalized structures that might enhance the concept of symmetry. Several requirements refer to constraints to the q- and r-factors or to the product of parameters itself. Our aim is to arrive at a generalization of the concept of group grading and Grassmann parameters. Thus, we will allow for indices such that $q_{\bar{a},\bar{a}} \neq 1$ as long as their parameters are nilpotent: $\beta_{\bar{a}}\beta_{\bar{a}} = 0$. We do not want any further reference to particular properties of the parameters, so we will constrain the further requirements *only* to the q- and r-factors. We determine some structures which will provide sufficient conditions for having gradings with respect to abelian groups.

Definition. We call ($\mathbb{I}$; q, r) a *group grading over* $\mathbb{K}$ *of order* N, with $N \geq 3$, iff $\mathbb{K}$ is a commutative field, $\{\mathbb{I}; +\}$ is an abelian group, $\tilde{o}$ the neutral element of $\mathbb{I}$; and q and r are applications

$$q\colon \quad \mathbb{I} \times \mathbb{I} \to \mathbb{K}\backslash\{0\}\ ; \qquad (\tilde{a}, \tilde{e}) \mapsto q(\tilde{a}, \tilde{e}) =: q_{\bar{a},\bar{e}} \tag{4.1}$$

$$r\colon \quad \mathbb{I} \times \mathbb{I} \times \mathbb{I} \to \mathbb{K}\backslash\{0\}\ ; \qquad (\tilde{a}, \tilde{e}, \tilde{c}) \mapsto r(\tilde{a}, \tilde{e}, \tilde{c}) =: r_{\bar{a},\bar{e},\bar{c}} \tag{4.2}$$

fulfilling the following requirements for all $\tilde{a}$, $\tilde{e}$, $\tilde{c}$ in $\mathbb{I}$:

$$q_{\bar{e},\bar{a}}q_{\bar{a},\bar{e}} = 1 \tag{4.3}$$

$$q_{\bar{a},\bar{o}} = 1 \tag{4.4}$$

$$r_{\bar{a},\bar{e},\bar{o}} = r_{\bar{a},\bar{o},\bar{e}} = r_{\bar{o},\bar{a},\bar{e}} = 1 \tag{4.5}$$

and the q- and r-coefficients fulfill all the consistency requirements obtained for parameter monomials up to order N, with the only exception of the consistency requirement (3.15), which will be fulfilled by requiring $q_{a,a} \neq 1 \Rightarrow \beta_a\beta_a = 0$.

Accordingly, the group gradings of order $N \geq 3$ will fulfill the requirements on q and r in (3.19):

$$q_{\tilde{a},\tilde{e}+\tilde{c}}\, q^{-1}_{\tilde{a},\tilde{e}}\, q^{-1}_{\tilde{a},\tilde{c}} = r_{\tilde{a},\tilde{e},\tilde{c}}\, r^{-1}_{\tilde{e},\tilde{a},\tilde{c}}\, r_{\tilde{e},\tilde{c},\tilde{a}} \tag{4.6}$$

$$q_{\tilde{a},\tilde{e}+\tilde{c}}\, q^{-1}_{\tilde{a},\tilde{e}}\, q^{-1}_{\tilde{c},\tilde{e}}\, q^{-1}_{\tilde{a}+\tilde{e},\tilde{c}} = r_{\tilde{a},\tilde{e},\tilde{c}}\, r_{\tilde{c},\tilde{e},\tilde{a}} \tag{4.7}$$

The group gradings of order $N \geq 4$ will additionally fulfill the requirements for the q- and r-factors given in (3.22) and (3.24), among others. The group gradings of order $N \geq 6$ will additionally fulfill the requirements for the q- and r-factors given in (3.26), among others.

The group grading of order N assures the consistency requirements for monomials up to order N. In order to provide a structure that allows for the definition of (I; q, r)-graded Lie algebras, we have already indicated that consistency requirements for monomials of order 6 have to be fulfilled. The aim is now the definition of a structure that provides consistency to the parameter monomials of *arbitrary order:*

Definition. We call (I; q, r) an *interactive group grading over* $\mathbb{K}$ iff $\mathbb{K}$ is a commutative field. {$\mathbb{K}$; +} is an abelian group, $\tilde{o}$ the additive neutral element of I; and there is an application R such that the q, r, and R applications

$$q\colon\ \mathrm{I} \times \mathrm{I} \to \mathbb{K}\backslash\{0\}; \qquad (\tilde{a}, \tilde{e}) \mapsto q(\tilde{a}, \tilde{e}) =: q_{\tilde{a},\tilde{e}} \tag{4.8}$$

$$r\colon\ \mathrm{I} \times \mathrm{I} \times \mathrm{I} \to \mathbb{K}\backslash\{0\}; \qquad (\tilde{a}, \tilde{e}, \tilde{c}) \mapsto r(\tilde{a}, \tilde{e}, \tilde{c}) =: r_{\tilde{a},\tilde{e},\tilde{c}} \tag{4.9}$$

$$R\colon\ \mathrm{I} \times \mathrm{I} \to \mathbb{K}\backslash\{0\}; \qquad (\tilde{a}, \tilde{e}) \mapsto R(\tilde{a}, \tilde{e}) =: R_{\tilde{a},\tilde{e}} \tag{4.10}$$

fulfill the following requirements for all $\tilde{a}$, $\tilde{e}$, $\tilde{c}$ in I (and for adequate and fixed choices of the phases in the roots):

$$q_{\tilde{e},\tilde{a}}\, q_{\tilde{a},\tilde{e}} = 1 \tag{4.11}$$

$$q_{\tilde{a},\tilde{o}} = 1 \tag{4.12}$$

$$r_{\tilde{a},\tilde{e},\tilde{o}} = r_{\tilde{a},\tilde{o},\tilde{e}} = r_{\tilde{o},\tilde{a},\tilde{e}} = 1 \tag{4.13}$$

$$q^{\text{“}1/2\text{”}}_{\tilde{a},\tilde{e}}\, R_{\tilde{a},\tilde{e}} = q_{\tilde{a},\tilde{a}}\, q^{\text{“}1/2\text{”}}_{\tilde{e},\tilde{a}}\, R_{\tilde{e},\tilde{a}} \tag{4.14}$$

$$q^{\text{“}1/2\text{”}}_{\tilde{a},\tilde{e}+\tilde{c}}\, R_{\tilde{a},\tilde{e}+\tilde{c}}\, q^{\text{“}1/2\text{”}}_{\tilde{e},\tilde{c}}\, R_{\tilde{e},\tilde{c}} = r_{\tilde{a},\tilde{e},\tilde{c}}\, q^{\text{“}1/2\text{”}}_{\tilde{a},\tilde{e}}\, R_{\tilde{a},\tilde{e}}\, q^{\text{“}1/2\text{”}}_{\tilde{a}+\tilde{e},\tilde{c}}\, R_{\tilde{a}+\tilde{e},\tilde{c}} \tag{4.15}$$

We will show that the latter are exactly the requirements for having consistency for parameter monomials of arbitrary order (i.e., when $N \to \infty$), as well as sufficient requirements for a closed graded Lie algebraic structure.

We introduce some useful adjectives for the defined group gradings over $\mathbb{K}$ for which particular properties are required:

Definition. A group grading $(\mathrm{I}; q, r)$ over $\mathbb{K}$ is called:

- *faithful* iff

$$\text{for all } \tilde{a}, \tilde{e} \in \mathrm{I}, \quad \tilde{a} \neq \tilde{e} \quad \exists \tilde{c}, \tilde{u} \in \mathrm{I} \quad (q_{a,c} \neq q_{e,c} \text{ or } r_{a,c,u} \neq r_{e,c,u} \text{ or } r_{c,a,u} \neq r_{c,e,u} \text{ or } r_{c,u,a} \neq r_{c,u,e}) \tag{4.16}$$

- *separated* iff the q-application fulfills

$$\text{for all} \quad \tilde{a}, \tilde{e}, \tilde{c} \in \mathrm{I} \quad (q_{\tilde{a},\tilde{e}+\tilde{c}} = q_{\tilde{a},\tilde{e}} q_{\tilde{a},\tilde{c}}) \tag{4.17}$$

- *associative* iff the r-application is constant:

$$\text{for all} \quad \tilde{a}, \tilde{e}, \tilde{c} \in \mathrm{I} \quad (r_{\tilde{a},\tilde{e},\tilde{c}} = 1) \tag{4.18}$$

- *commutative* iff the q-application is constant:

$$\text{for all} \quad \tilde{a}, \tilde{e} \in \mathrm{I} \quad (q_{\tilde{a},\tilde{e}} = 1) \tag{4.19}$$

- *trivial* iff it is commutative and associative
- *with involution* iff there exist involutions $(\cdot)^\star$ and $(\cdot)^*$ in I and $\mathbb{K}$, respectively, such that for all $\tilde{a}$ $\tilde{e}$, $\tilde{c}$ in I

$$q_{\tilde{a}^\star,\tilde{e}^\star} = (q^*_{\tilde{a},\tilde{e}})^{-1} \tag{4.20}$$

$$r_{\tilde{c}^\star,\tilde{e}^\star,\tilde{a}^\star} = (r^*_{\tilde{a},\tilde{e},\tilde{c}})^{-1} \tag{4.21}$$

- *extending group grading over* $\mathbb{K}$ *of* $(\mathrm{I}'; q', r')$ iff $(\mathrm{I}'; q', r')$ is a trivial group grading over $\mathbb{K}$, $(\mathrm{I}; q, r)$ is a faithful iterative group grading over $\mathbb{K}$, and

$$\mathrm{I}' \subset \mathrm{I}, \qquad q' = q|_{\mathrm{I}'\times\mathrm{I}'}, \qquad r' = r|_{\mathrm{I}'\times\mathrm{I}'\times\mathrm{I}'} \tag{4.22}$$

Observe that every associative and every commutative group grading is separated. The separated group gradings of order $N \geq 3$ fulfill [see (4.3), (4.4), (4.6), (4.7), and (4.17)]

$$(q_{\tilde{a},\tilde{e}} = q^{-1}_{-\tilde{a},\tilde{e}} = q^{-1}_{\tilde{a},-\tilde{e}} = q_{-a,-\tilde{e}}) \qquad \text{for all} \quad \tilde{a}, \tilde{e} \in \mathrm{I} \tag{4.23}$$

$$(r_{a,\tilde{e},\tilde{c}} = r^{-1}_{\tilde{c},\tilde{e},\tilde{a}}) \qquad \text{for all} \quad \tilde{a}, \tilde{e}, \tilde{c} \in \mathrm{I} \tag{4.24}$$

$$(r_{\tilde{a},\tilde{e},\tilde{c}} r_{\tilde{c},\tilde{a},\tilde{e}} r_{\tilde{e},\tilde{c},\tilde{a}} = 1) \qquad \text{for all} \quad \tilde{a}, \tilde{e}, \tilde{c} \in \mathrm{I} \tag{4.25}$$

Hence, the interchange of the first and last arguments in the r-factors of separated group gradings provides the multiplicative inverse, and the product of the r-factors with cycled permuted arguments gives the identity.

Table I. Z_2-Group.

$(+)^{Z_2}$	0	1
0	0	1
1	1	0

We will define now particular group gradings over C. The most elementary group grading beyond the trivial group grading is the group grading of the Grassmann algebras:

Definition. We call $(Z_2; q^{Z_2}; r^{Z_2})$ a *supergrading or a Grassmann* grading over C (or R) iff

$$\{Z_2; +\}; \quad Z_2 = \{0, 1\} \tag{4.26}$$

$$+: \quad Z_2 \times Z_2 \to Z_2; \quad (a_o, e_o) \mapsto (a_o + e_o) \bmod 2 \tag{4.27}$$

$$q^{Z_2}: \quad Z_2 \times Z_2 \to C; \quad (a_o, e_o) \mapsto \exp\{i\pi a_o e_o\} \tag{4.28}$$

$$r^{Z_2}: \quad Z_2 \times Z_2 \times Z_2 \to C; \quad (a_o, e_o, c_o) \mapsto 1 \tag{4.29}$$

The addition and the q^{Z_2} applications can be expressed through Tables I and II (where the first argument is read from the first column and the second from the first line as usual).

We observe from the Table II that q^{Z_2} is a symmetric application. It is simple to verify that $(Z_2; q^{Z_2}, r^{Z_2})$ is a separated, associative, faithful, iterative group grading over C (or R).

Proposition 0. There are no non-associative group gradings for the Z_2 group (4.26)–(4.27) and the q-application (4.28).

Proof. $\tilde{o}$ is the neutral element of Z_2. So $r_{\tilde{o},\tilde{a},\tilde{e}} = r_{\tilde{a},\tilde{o},\tilde{e}} = r_{\tilde{a},\tilde{e},\tilde{o}} = 1$ for all $\tilde{a},\tilde{e} \in Z_2$. Hence, the only possible nontrivial r-factor would be $r_{1,1,1}$. Now, the property (4.17) holds for the q-factors in (4.28), so (4.24) and (4.25) hold as well. Hence $r_{1,1,1} = r^{-1}_{1,1,1}$ and $r_{1,1,1}r_{1,1,1}r_{1,1,1} = 1$. The only possible solution is $r_{1,1,1} = 1$. ■

The associative group gradings involving *symmetric* q-applications $q_{\tilde{a},\tilde{e}} = q_{\tilde{e},\tilde{a}} = q^{-1}_{\tilde{a},\tilde{e}}$ imply $q_{\tilde{a},\tilde{e}} = 1$ or $q_{\tilde{a},\tilde{e}} = -1$. The commutation relations of

Table II. q^{Z_2}-factors

q^{Z_2}	0	1
0	1	1
1	1	−1

quantum fields using such group gradings are shown to be equivalent to supergradings (called "normal commutation relations") or replications of them by using the so-called Klein transformations (Klein, 1938; Araki, 1961). This exploits the existence of superselection rules.

In order to have more chance to arrive at structures which can be "essentially unequivalent" to supergradings, we ask for *nonsymmetric q-applications*. We recall first some identities of R^3-geometry. Let $\vec{\alpha}, \vec{A}, \vec{E}, \vec{C} \in R^3$ and let "·" and "×" be the standard scalar and vector products of R^3-vectors respectively. Then

$$\vec{\alpha} \cdot \{\vec{A} \times \vec{E}\} = -\vec{\alpha} \cdot \{\vec{E} \times \vec{A}\} \tag{4.30}$$

$$\vec{\alpha} \cdot \{\vec{A} \times (\vec{E} + \vec{C})\} = \vec{\alpha} \cdot \{\vec{A} \times \vec{E}\} + \vec{\alpha} \cdot \{\vec{A} \times \vec{C}\} \tag{4.31}$$

We observe that a q-application of the form

$$q_{[\vec{A}],[\vec{E}]} = \exp\{\vec{\alpha} \cdot \{\vec{A} \times \vec{E}\}\} \tag{4.32}$$

with $\vec{\alpha}$ or $i\vec{\alpha}$ a fixed vector in R^3, will fulfill the requirements (4.3) and (4.17) if $[\vec{A} + \vec{E}] = [\vec{A}] + [\vec{e}]$. The requirement (4.4) is fulfilled adopting $\tilde{o} = [\vec{0}]$.

We can add to this construction three different models for the r-factors:

$$r^{(\text{null})}_{[\vec{A}],[\vec{E}],[\vec{C}]} = 1 \tag{4.33}$$

The choice (4.32) and (4.33) fulfills the equations (4.11)–(4.13), (4.17), and fulfills (4.14)–(4.15) for $R_{a,\bar{c}} := 1$.

A further and less trivial choice of r-application is given by:

$$r^{(\triangle)}_{[\vec{A}],[\vec{E}],[\vec{C}]} = \exp\{\vec{\rho} \cdot \{\vec{E} \times (\vec{A} \times \vec{C})\}\} \tag{4.34}$$

with $\vec{\rho}$ or $i\vec{\rho}$ a fixed vector in R^3. The choice (4.32) and (4.34) fulfills the equations (4.11)–(4.13), (4.17), and fulfills (4.14)–(4.15) for $R_{a,\bar{c}} := (r_{a,\bar{c}-a,\bar{c}})^{1/3}$.

A further choice of r-application is given by

$$r^{(\square)}_{[\vec{A}],[\vec{E}],[\vec{C}]} = \exp\{k(\vec{A} \times \vec{C}) \cdot \{(\vec{A} + \vec{C}) \times \vec{E}\}\} \tag{4.35}$$

with k or ik a fixed element of R. The choice (4.32) and (4.35) fulfills the equations (4.11)–(4.13), (4.17), and fulfills (4.14)–(4.15) for $R_{a,\bar{c}} := (r_{a,\bar{c}-a,\bar{c}})^{1/4}$.

These observations lead to the following structure:

Definition. We call (G, q^G, r^G) an *axial grading over* C of *null, cubic, or quartic type* iff:

- {G; +} abelian group.
- The elements of G are equivalence classes $[\vec{A}]$, $[\vec{E}]$, ... of vectors in R^3, where $[\vec{A}] + [\vec{E}] = [\vec{A} + \vec{E}]$. G has neutral element $\tilde{o} = [\vec{0}]$.

- The application q^G takes the form (4.32), with $\vec{\alpha}$ or $i\vec{\alpha}$ a fixed vector in R^3.
- The application r^G takes for the null, cubic, and quartic type, respectively, the form (4.33), the form (4.34) (with $\vec{\rho}$ or $i\vec{\rho}$ a fixed vector in R^3), and the form (4.35) (with k or ik a fixed number in R).
- The applications q^G and r^G are well defined, i.e., they are independent of the choice of class representatives $\vec{A}$, $\vec{E}$, $\vec{C}$, . . . , from the classes $[\vec{A}]$, $[\vec{E}]$, $[\vec{C}]$, . . . , respectively, in the definitions of q^G and r^G.

It is easy to verify that every axial grading over C is a separated iterative group grading over C.

We can combine group gradings over a field K by the following recipe.

Definition. We call $(I \times I'; q^{I\times I'}, r^{I\times I'})$ a *direct product* group grading over K iff $(I; q^I, r^I)$ and $(I'; q^{I'}, r^{I'})$ are both group gradings over K, and the applications $q^{I\times I'}$ and $r^{I\times I'}$ are given by

$$q^{I\times I'}_{(\tilde{a},\tilde{a}'),(\tilde{e},\tilde{e}')} = q^I_{\tilde{a},\tilde{e}} q^{I'}_{\tilde{a}',\tilde{e}'} \tag{4.36}$$

$$r^{I\times I'}_{(\tilde{a},\tilde{a}')(\tilde{e},\tilde{e}')(\tilde{c},\tilde{c}')} = r^I_{\tilde{a},\tilde{e},\tilde{c}} r^{I'}_{\tilde{a}',\tilde{e}',\tilde{c}'} \tag{4.37}$$

Definition. The direct products between a supergrading and axial gradings over C are group gradings over C called *single gradings* over C.

Proposition 1. Every super, axial, and single grading over C is a separated iterative group grading. This can be easily proved verifying all the properties of separated iterative group gradings.

We finish this section by determining some remarkable subsets of the grading group I of a group grading (I; q, r) over C. According to the so-called "small Fermat theorem," in every finite group $\{I; +\}$ of order $N(I)$ and neutral element $\tilde{o}$ we have $\tilde{a} + \ldots + \tilde{a}$ (summation $N(I)$ times) $= \tilde{o}$ for all $\tilde{o}$ in I. It is meaningful to consider the following subsets of the abelian group I (either finite or not):

$$I^{(1/2)} = \{\tilde{a} \in I\colon \tilde{a} + \tilde{a} = \tilde{o}\} \tag{4.38}$$

$$I^{(1/n)} = \{\tilde{a} \in I\colon \tilde{a} + \cdots + \tilde{a}\ (n \text{ times}) = \tilde{o}\}, \qquad n \in N \tag{4.39}$$

It is easy to verify

$$\{I^{(1/n)}; +\} \text{ abelian subgroup of } \{I; +\} \tag{4.40}$$

Since $q_{\tilde{a},\tilde{a}} = q^{-1}_{\tilde{a},\tilde{a}}$, then either $q_{\tilde{a},\tilde{a}} = 1$ or $q_{\tilde{a},\tilde{a}} = -1$. We can thus divide $I^{(1/2)}$ into the two disjoint subsets $I^{(+1/2)}$ and $I^{(-1/2)}$:

$$I^{(\pm 1/2)} = \{\tilde{a} \in I\colon \tilde{a} + \tilde{a} = \tilde{o} \quad \text{and} \quad q_{\tilde{a},\tilde{a}} = \pm 1\} \tag{4.41}$$

$$I^{(1/2)} = I^{(+1/2)} \cup I^{(-1/2)}, \qquad I^{(+1/2)} \cap I^{(-1/2)} = \emptyset \tag{4.42}$$

In the case in which $(\mathbb{I}; q, r)$ is a separated group grading we can easily see that (4.17) implies

$$\{\mathbb{I}^{(+1/2)}; +\} \text{ abelian group when } (\mathbb{I}; q, r) \text{ separated} \tag{4.43}$$

For $(\mathbb{I}; q, r)$ separated we observe also that $\tilde{a} \in \mathbb{I}^{(1/2)}$ implies $\tilde{a} = -\tilde{a}$ and thus for all $\tilde{e} \in \mathbb{I}$ $(q_{\tilde{a},\tilde{e}} = q_{\tilde{a},\tilde{e}}^{-1})$. Hence,

$$(\tilde{a} \in \mathbb{I}^{(1/2)} \text{ and } \tilde{e} \in \mathbb{I} \Rightarrow q_{\tilde{a},\tilde{e}} \in \{+1, -1\}) \qquad \text{for all} \quad (\mathbb{I}; q, r) \text{ separated} \tag{4.44}$$

We consider now the following subset of $\mathbb{I}^{(+1/2)}$:

$$\mathbb{I}^{(++1/2)} = \{\tilde{a} \in \mathbb{I}^{(+1/2)} : \forall \tilde{e}, \tilde{c} \in \mathbb{I}^{(+1/2)}(q_{\tilde{a},\tilde{e}} = 1 \text{ and } r_{\tilde{a},\tilde{e},\tilde{c}} = 1)\} \tag{4.45}$$

The set $\mathbb{I}^{(++1/2)}$ fulfills

$$\{\mathbb{I}^{(++1/2)}; +\} \text{ abelian subgroup of } \{\mathbb{I}; +\} \text{ for } (\mathbb{I}; q, r) \text{ separated} \tag{4.46}$$

$$(\mathbb{I}^{(++1/2)}; q|_{\mathbb{I}^{(++1/2)2}}, r|_{\mathbb{I}^{(++1/2)3}}) \text{ trivial group grading over } \mathbb{K} \tag{4.47}$$

Hence, given $(\mathbb{I}; q, r)$ a faithful separated iterative group grading, then it is an *extending group grading* of the group grading (4.47). This observation is to be exploited for the construction of the extending group gradings for the Lie algebras (Wills-Toro, 1995).

5. EXAMPLES OF GROUP GRADINGS OVER $\mathbb{C}$ OR $\mathbb{R}$

We list now several group gradings over $\mathbb{C}$ or $\mathbb{R}$ which are of interest in mathematical physics.

Example 1. We introduce some definitions toward the construction of axial gradings over $\mathbb{C}$:

$$\{\mathbb{Z}_{4\Lambda} \times \mathbb{Z}_{4\Lambda}; +\}; \qquad \Lambda \in \mathbb{N} \text{ fixed}; \qquad \tilde{o} = (0, 0) \tag{5.1}$$

$$\mathbb{Z}_{4\Lambda} \times \mathbb{Z}_{4\Lambda} = \{(n, m): n, m = 0, 1, \ldots, 4\Lambda - 1\} \tag{5.2}$$

$$(n, m) + (n', m') = ((n + n') \bmod 4\Lambda, (m + m') \bmod 4\Lambda) \tag{5.3}$$

$$q^{\mathbb{Z}_{4\Lambda}\times\mathbb{Z}_{4\Lambda}}_{(n,m),(n',m')} = e^{(i\pi/2\Lambda)(nm'-mn')} \tag{5.4}$$

$$r^{\mathbb{Z}_{4\Lambda}\times\mathbb{Z}_{4\Lambda}}_{(n,m),(n',m'),(n'',m'')} = e^{(i\pi/2\Lambda)(nm''-mn'')[(n+n'')m'-n'(m+m')]} \tag{5.5}$$

$$r^{(\text{null})}_{(n,m),(n',m'),(n'',m'')} = 1 \tag{5.6}$$

We find the following faithful axial iterative group gradings over $\mathbb{C}$:

$$(\mathbb{Z}_{4\Lambda} \times \mathbb{Z}_{4\Lambda};\, q^{\mathbb{Z}_{4\Lambda}\times\mathbb{Z}_{4\Lambda}},\, r^{\mathbb{Z}_{4\Lambda}\times\mathbb{Z}_{4\Lambda}}) \qquad \text{not associative} \tag{5.7}$$

$$(\mathbb{Z}_{4\Lambda} \times \mathbb{Z}_{4\Lambda};\, q^{\mathbb{Z}_{4\Lambda}\times\mathbb{Z}_{4\Lambda}},\, r^{(\text{null})}) \qquad \text{associative} \tag{5.8}$$

The axial gradings (5.7) and (5.8) are, respectively, of quartic and null type. Using the definitions (4.38), (4.41), and (4.45), we obtain

$$\begin{aligned}(\mathbb{Z}_{4\Lambda} \times \mathbb{Z}_{4\Lambda})^{(1/2)} &= (\mathbb{Z}_{4\Lambda} \times \mathbb{Z}_{4\Lambda})^{(++1/2)} \\ &= \{(2\Lambda n, 2\Lambda m)\colon n, m = 0, 1\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2\end{aligned} \tag{5.9}$$

Accordingly, the group gradings (5.7) and (5.8) are extending group gradings over C of the trivial group grading of $\mathbb{Z}_2 \times \mathbb{Z}_2$. $\mathbb{Z}_2 \times \mathbb{Z}_2$ is precisely a suitable grading for the Poincaré algebra, and is also the set of spacetime discrete symmetries $L/L_+^\uparrow$, where L is the Lorentz group and $L_+^\uparrow$ is the proper orthochronous invariant subgroup of L.

In particular, for $\Lambda = 1$ we obtain the group grading for $\mathbb{Z}_4 \times \mathbb{Z}_4$ with the addition table shown in Table III. The values of the $q^{\mathbb{Z}_4\times\mathbb{Z}_4}$-application are presented in Table IV.

Example 2. From the direct product of the $(\mathbb{Z}_2;\, q^{\mathbb{Z}_2},\, r^{\mathbb{Z}_2})$ supergrading and the axial iterative group gradings of Example 1 we obtain the following faithful iterative single group gradings over C:

Table III. Addition Table of the Group $\mathbb{Z}_4 \times \mathbb{Z}_4$

$+$	$\tilde{o}\ \tilde{u}^1\ \tilde{u}^2\ \tilde{u}^3$	$\tilde{e}^0\ \tilde{e}^1\ \tilde{e}^2\ \tilde{e}^3$	$\tilde{c}^0\ \tilde{c}^1\ \tilde{c}^2\ \tilde{c}^3$	$\tilde{s}^0\ \tilde{s}^1\ \tilde{s}^2\ \tilde{s}^3$
$(0,0) = \tilde{o}$	$\tilde{o}\ \tilde{u}^1\ \tilde{u}^2\ \tilde{u}^3$	$\tilde{e}^0\ \tilde{e}^1\ \tilde{e}^2\ \tilde{e}^3$	$\tilde{c}^0\ \tilde{c}^1\ \tilde{c}^2\ \tilde{c}^3$	$\tilde{s}^0\ \tilde{s}^1\ \tilde{s}^2\ \tilde{s}^3$
$(2,0) = \tilde{u}^1$	$\tilde{u}^1\ \tilde{o}\ \tilde{u}^3\ \tilde{u}^2$	$\tilde{e}^1\ \tilde{e}^0\ \tilde{e}^3\ \tilde{e}^2$	$\tilde{c}^3\ \tilde{c}^2\ \tilde{c}^1\ \tilde{c}^0$	$\tilde{s}^2\ \tilde{s}^3\ \tilde{s}^0\ \tilde{s}^1$
$(0,2) = \tilde{u}^2$	$\tilde{u}^2\ \tilde{u}^3\ \tilde{o}\ \tilde{u}^1$	$\tilde{e}^2\ \tilde{e}^3\ \tilde{e}^0\ \tilde{e}^1$	$\tilde{s}^1\ \tilde{s}^0\ \tilde{s}^3\ \tilde{s}^2$	$\tilde{s}^3\ \tilde{s}^2\ \tilde{s}^1\ \tilde{s}^0$
$(2,2) = \tilde{u}^3$	$\tilde{u}^3\ \tilde{u}^2\ \tilde{u}^1\ \tilde{o}$	$\tilde{e}^3\ \tilde{e}^2\ \tilde{e}^1\ \tilde{e}^0$	$\tilde{c}^2\ \tilde{c}^3\ \tilde{c}^0\ \tilde{c}^1$	$\tilde{s}^1\ \tilde{s}^0\ \tilde{s}^3\ \tilde{s}^2$
$(1,0) = \tilde{e}^0$	$\tilde{e}^0\ \tilde{e}^1\ \tilde{e}^2\ \tilde{e}^3$	$\tilde{u}^1\ \tilde{o}\ \tilde{u}^3\ \tilde{u}^2$	$\tilde{s}^1\ \tilde{s}^2\ \tilde{s}^0\ \tilde{s}^3$	$\tilde{c}^1\ \tilde{c}^3\ \tilde{c}^2\ \tilde{c}^0$
$(3,0) = \tilde{e}^1$	$\tilde{e}^1\ \tilde{e}^0\ \tilde{e}^3\ \tilde{e}^2$	$\tilde{o}\ \tilde{u}^1\ \tilde{u}^2\ \tilde{u}^3$	$\tilde{s}^3\ \tilde{s}^0\ \tilde{s}^2\ \tilde{s}^1$	$\tilde{c}^2\ \tilde{c}^0\ \tilde{c}^1\ \tilde{c}^3$
$(1,2) = \tilde{e}^2$	$\tilde{e}^2\ \tilde{e}^3\ \tilde{e}^0\ \tilde{e}^1$	$\tilde{u}^3\ \tilde{u}^2\ \tilde{u}^1\ \tilde{o}$	$\tilde{s}^2\ \tilde{s}^1\ \tilde{s}^3\ \tilde{s}^0$	$\tilde{c}^0\ \tilde{c}^2\ \tilde{c}^3\ \tilde{c}^1$
$(3,2) = \tilde{e}^3$	$\tilde{e}^3\ \tilde{e}^2\ \tilde{e}^1\ \tilde{e}^0$	$\tilde{u}^2\ \tilde{u}^3\ \tilde{o}\ \tilde{u}^1$	$\tilde{s}^0\ \tilde{s}^3\ \tilde{s}^1\ \tilde{s}^2$	$\tilde{c}^3\ \tilde{c}^1\ \tilde{c}^0\ \tilde{c}^2$
$(0,1) = \tilde{c}^0$	$\tilde{c}^0\ \tilde{c}^3\ \tilde{c}^1\ \tilde{c}^2$	$\tilde{s}^1\ \tilde{s}^3\ \tilde{s}^2\ \tilde{s}^0$	$\tilde{u}^2\ \tilde{o}\ \tilde{u}^1\ \tilde{u}^3$	$\tilde{e}^1\ \tilde{e}^2\ \tilde{e}^0\ \tilde{e}^3$
$(0,3) = \tilde{c}^1$	$\tilde{c}^1\ \tilde{c}^2\ \tilde{c}^0\ \tilde{c}^3$	$\tilde{s}^2\ \tilde{s}^0\ \tilde{s}^1\ \tilde{s}^3$	$\tilde{o}\ \tilde{u}^2\ \tilde{u}^3\ \tilde{u}^1$	$\tilde{e}^3\ \tilde{e}^0\ \tilde{e}^2\ \tilde{e}^1$
$(2,3) = \tilde{c}^2$	$\tilde{c}^2\ \tilde{c}^1\ \tilde{c}^3\ \tilde{c}^0$	$\tilde{s}^0\ \tilde{s}^2\ \tilde{s}^3\ \tilde{s}^1$	$\tilde{u}^1\ \tilde{u}^3\ \tilde{u}^2\ \tilde{o}$	$\tilde{e}^2\ \tilde{e}^1\ \tilde{e}^3\ \tilde{e}^0$
$(2,1) = \tilde{c}^3$	$\tilde{c}^3\ \tilde{c}^0\ \tilde{c}^2\ \tilde{c}^1$	$\tilde{s}^3\ \tilde{s}^1\ \tilde{s}^0\ \tilde{s}^2$	$\tilde{u}^3\ \tilde{u}^1\ \tilde{o}\ \tilde{u}^2$	$\tilde{e}^0\ \tilde{e}^3\ \tilde{e}^1\ \tilde{e}^2$
$(3,3) = \tilde{s}^0$	$\tilde{s}^0\ \tilde{s}^2\ \tilde{s}^3\ \tilde{s}^1$	$\tilde{c}^1\ \tilde{c}^2\ \tilde{c}^0\ \tilde{c}^3$	$\tilde{e}^1\ \tilde{e}^3\ \tilde{e}^2\ \tilde{e}^0$	$\tilde{u}^3\ \tilde{o}\ \tilde{u}^2\ \tilde{u}^1$
$(1,1) = \tilde{s}^1$	$\tilde{s}^1\ \tilde{s}^3\ \tilde{s}^2\ \tilde{s}^0$	$\tilde{c}^3\ \tilde{c}^0\ \tilde{c}^2\ \tilde{c}^1$	$\tilde{e}^2\ \tilde{e}^0\ \tilde{e}^1\ \tilde{e}^3$	$\tilde{o}\ \tilde{u}^3\ \tilde{u}^1\ \tilde{u}^2$
$(1,3) = \tilde{s}^2$	$\tilde{s}^2\ \tilde{s}^0\ \tilde{s}^1\ \tilde{s}^3$	$\tilde{c}^3\ \tilde{c}^0\ \tilde{c}^2\ \tilde{c}^1$	$\tilde{e}^0\ \tilde{e}^2\ \tilde{e}^3\ \tilde{e}^1$	$\tilde{u}^2\ \tilde{u}^1\ \tilde{u}^3\ \tilde{o}$
$(3,1) = \tilde{s}^3$	$\tilde{s}^3\ \tilde{s}^1\ \tilde{s}^0\ \tilde{s}^2$	$\tilde{c}^0\ \tilde{c}^3\ \tilde{c}^1\ \tilde{c}^2$	$\tilde{e}^3\ \tilde{e}^1\ \tilde{e}^0\ \tilde{e}^2$	$\tilde{u}^1\ \tilde{u}^2\ \tilde{o}\ \tilde{u}^3$

Table IV. $q^{\mathbb{Z}_4\times\mathbb{Z}_4}$-Factors

$q^{\mathbb{Z}_4\times\mathbb{Z}_4}$	$\tilde{o}$	$\tilde{u}^1$	$\tilde{u}^2$	$\tilde{u}^3$	$\tilde{e}^0$	$\tilde{e}^1$	$\tilde{e}^2$	$\tilde{e}^3$	$\tilde{c}^0$	$\tilde{c}^1$	$\tilde{c}^2$	$\tilde{c}^3$	$\tilde{s}^0$	$\tilde{s}^1$	$\tilde{s}^2$	$\tilde{s}^3$
$(0, 0) = \tilde{o}$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$(2, 0) = \tilde{u}^1$	1	1	1	1	1	1	1	1	−1	−1	−1	−1	−1	−1	−1	−1
$(0, 2) = \tilde{u}^2$	1	1	1	1	−1	−1	−1	−1	1	1	1	1	−1	−1	−1	−1
$(2, 2) = \tilde{u}^3$	1	1	1	1	−1	−1	−1	−1	−1	−1	−1	−1	1	1	1	1
$(1, 0) = \tilde{e}^0$	1	1	−1	−1	1	1	−1	−1	i	−i	−i	i	−i	i	−i	i
$(3, 0) = \tilde{e}^1$	1	1	−1	−1	1	1	−1	−1	−i	i	i	−i	i	−i	i	−i
$(1, 2) = \tilde{e}^2$	1	1	−1	−1	−1	−1	1	1	i	−i	−i	i	i	−i	i	−i
$(3, 2) = \tilde{e}^3$	1	1	−1	−1	−1	−1	1	1	−i	i	i	−i	−i	i	−i	i
$(0, 1) = \tilde{c}^0$	1	−1	1	−1	−i	i	−i	i	1	1	−1	−1	i	−i	−i	i
$(0, 3) = \tilde{c}^1$	1	−1	1	−1	i	−i	i	−i	1	1	−1	−1	−i	i	i	−i
$(2, 3) = \tilde{c}^2$	1	−1	1	−1	i	−i	i	−i	−1	−1	1	1	i	−i	−i	i
$(2, 1) = \tilde{c}^3$	1	−1	1	−1	−i	i	−i	i	−1	−1	1	1	−i	i	i	−i
$(3, 3) = \tilde{s}^0$	1	−1	−1	1	i	−i	−i	i	−i	i	−i	i	1	1	−1	−1
$(1, 1) = \tilde{s}^1$	1	−1	−1	1	−i	i	i	−i	i	−i	i	−i	1	1	−1	−1
$(1, 3) = \tilde{s}^2$	1	−1	−1	1	i	−i	−i	i	i	−i	i	−i	−1	−1	1	1
$(3, 1) = \tilde{s}^3$	1	−1	−1	1	−i	i	i	−i	−i	i	−i	i	−1	−1	1	1

$$(\mathbb{Z}_2 \times (\mathbb{Z}_{4\Lambda} \times \mathbb{Z}_{4\Lambda}); q^{\mathbb{Z}_2} q^{\mathbb{Z}_{4\Lambda}\times\mathbb{Z}_{4\Lambda}}, r^{\mathbb{Z}_2} r^{\mathbb{Z}_{4\Lambda}\times\mathbb{Z}_{4\Lambda}}) \qquad \text{not associative} \tag{5.10}$$

$$(\mathbb{Z}_2 \times (\mathbb{Z}_{4\Lambda} \times \mathbb{Z}_{4\Lambda}); q^{\mathbb{Z}_2} q^{\mathbb{Z}_{4\Lambda}\times\mathbb{Z}_{4\Lambda}}, r^{\mathbb{Z}_2} r^{(\text{null})})] \qquad \text{associative} \tag{5.11}$$

We find as well

$$(\mathbb{Z}_2 \times (\mathbb{Z}_{4\Lambda} \times \mathbb{Z}_{4\Lambda}))^{(++1/2)} = \{0\} \times (\mathbb{Z}_{4\Lambda} \times \mathbb{Z}_{4\Lambda})^{(++1/2)} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \tag{5.12}$$

Accordingly, the group gradings (5.10) and (5.11) provide extending group gradings over C of the trivial group grading $\mathbb{Z}_2 \times \mathbb{Z}_2$. They provide (Wills-Toro, 1995) serious candidates for the graded extensions of the Poincaré algebra (special relativity) beyond supersymmetry.

6. GROUP GRADED ALGEBRAS

In the previous sections, we have articulated the group grading structures which fulfill several conditions suitable for the definition of I-graded, q-commutative, and r-associative parameters. Nevertheless, the requirements explored only consistency conditions up to parameter monomials of order six. We are not sure so far if further independent conditions shall be necessary for the consistency of monomials of arbitrary order. We will determine sufficient conditions for such consistency.

We define graded algebras instead of plain algebras since the addition operation is not defined for the whole algebra. The addition operation is defined only in certain subsets.

Definition. We call $\{P; +, \bullet, \cdot\}$ an *(I; q, r)-graded algebra* over $\mathbb{K}$ iff $P \neq \{0\}$ is a set whose elements are called *parameters*, $0 \in P$, and $(I; q, r)$ is a *group grading* over $\mathbb{K}$ fulfilling the following three axioms:

Axiom 1. There is an application S_t that assigns an element (tilded index) of I to each element of $P\backslash\{0\}$. The abelian group I is generated by the image of $P\backslash\{0\}$ under S_t. The set of $P_{\tilde{a}}$ of preimages for each $\tilde{a} \in I$ with the null element 0 constitutes a vector space over $\mathbb{K}$:

$$S_t\colon \quad P\backslash\{0\} \to I, \qquad \beta_a \mapsto S_t(\beta_a) = \tilde{a} \tag{6.1}$$

$$\text{I is generated by } S_t(P\backslash\{0\}), \tag{6.2}$$

$$P_{\tilde{a}} = \{0\} \cup S_t^{-1}(\tilde{a}), \qquad \tilde{a} \in I \tag{6.3}$$

$$\{P_{\tilde{a}}; +, \cdot\} \text{ vector space over } \mathbb{K}, \qquad \tilde{a} \in I \tag{6.4}$$

$$P = \bigcup_{\tilde{a} \in I} P_{\tilde{a}} \neq \{0\} \tag{6.5}$$

Axiom 2. The set P with its product "$\bullet$" of parameters builds a closed I-graded, q-commutative, and r-associative structure, i.e., for all $\beta_a, \beta'_e, \beta''_c \in P$:

$$\bullet\colon \quad P \times P \to P; \qquad (\beta_a, \beta'_e) \mapsto \beta_a\beta'_e \in P \qquad \text{(closure)} \tag{6.6}$$

$$P_{\tilde{a}} \bullet P_{\tilde{e}} \subset P_{\tilde{a}+\tilde{e}}, \qquad \tilde{a}, \tilde{e} \in I \qquad \text{(I-grading)} \tag{6.7}$$

$$\beta_a\beta'_e = q_{a,e}\beta'_e\beta_a \qquad (q\text{-commutativity}) \tag{6.8}$$

$$\beta_a(\beta'_e\beta''_c) = r_{a,e,c}(\beta_a\beta'_e)\beta''_c \qquad (r\text{-associativity}) \tag{6.9}$$

Axiom 3. The product operation "$\bullet$" of parameters is bilinear with respect to the operations (addition and product by a scalar of $\mathbb{K}$) defined in each vector space $P_{\tilde{a}}$, $\tilde{a} \in I$, i.e., for all $\beta_{\tilde{a}}, \beta''_{\tilde{a}}, \beta'_{\tilde{e}} \in P$, and for all $y \in \mathbb{K}$,

$$(\beta_a + y\beta''_a)\beta'_e = \beta_a\beta'_e + y\beta''_a\beta'_e \tag{6.10}$$

$$\beta'_e(\beta_a + y\beta''_a) = \beta'_e\beta_a + y\beta'_e\beta''_a \qquad \text{(bilinearity)} \tag{6.11}$$

We could consider additionally the existence of an involution operation in P. This leads to the further definition:

Definition. We call $\{P; +, \bullet, \cdot\}$ an (I; q, r)-graded algebra over $\mathbb{K}$ *with involution* $\overline{(\cdot)}$ iff additionally (I; q, r) is a group grading over $\mathbb{K}$ with involution, and the involutions $\overline{(\cdot)}$, $(\cdot)^{\star}$, $(\cdot)^*$ in $\mathbb{P}$, $\mathbb{I}$, $\mathbb{K}$, respectively, fulfill for all $\beta_{\tilde{a}}$, $\beta'_{\tilde{e}} \in \mathbb{P}$ and $y \in \mathbb{K}$

$$\overline{(\cdot)}: \quad \mathbb{P} \to \mathbb{P}, \qquad y\beta^{a} \mapsto y^*\overline{\beta}_{\tilde{a}^{\star}} \tag{6.12}$$

$$\overline{(y^*\overline{\beta}_{\tilde{a}^{\star}})} = y\beta_{\tilde{a}}, \qquad \overline{(y\beta_{\tilde{a}}\beta'_{\tilde{e}})} = y^*\overline{\beta}'_{\tilde{e}^{\star}}\overline{\beta}_{\tilde{a}^{\star}} \tag{6.13}$$

Definition. We call $\{\mathbb{P}; +, \bullet, \cdot\}$, respectively, *an iterative, a separated, a faithful, an associative, a commutative, a trivial, a super, an axial, a single* $(\mathbb{I}; q, r)$-graded algebra over $\mathbb{K}$ iff $(\mathbb{I}; q, r)$ is, respectively, an iterative, a separated, a faithful, an associative, a commutative, a trivial, a super, an axial, a single group grading over $\mathbb{K}$.

Definition. We call $\{\mathbb{P}^1 \times \mathbb{P}^2; +, \bullet, \cdot\}$ a *direct product* $(\mathbb{I}^1 \times \mathbb{I}^2; q^{\mathbb{I}^1}q^{\mathbb{I}^2}, r^{\mathbb{I}^1}r^{\mathbb{I}^2})$-graded algebra over $\mathbb{K}$ iff

(i) $\{\mathbb{P}^i; +, \bullet, \cdot\}$ is an $(\mathbb{I}^i; q^{\mathbb{I}^i}, r^{\mathbb{I}^i})$-graded algebra over $\mathbb{K}$, $i = 1, 2$.

(ii) $(\mathbb{I}^1 \times \mathbb{I}^2; q^{\mathbb{I}^1}q^{\mathbb{I}^2}, r^{\mathbb{I}^1}r^{\mathbb{I}^2})$ is the direct product group grading between $(\mathbb{I}^1; q^{\mathbb{I}^1}, r^{\mathbb{I}^1})$ and $(\mathbb{I}^2; q^{\mathbb{I}^2}, r^{\mathbb{I}^2})$.

(iii) The operations $+, \bullet, \cdot$ in $\{\mathbb{P}^1 \times \mathbb{P}^2; +, \bullet, \cdot\}$ are the naive extensions of the operations in $\mathbb{P}^1 \times \mathbb{P}^2$:

$$S_t((\beta^1_{\tilde{a}^1}, \beta^2_{\tilde{a}^2})) = (\tilde{a}^1, \tilde{a}^2) \in \mathbb{I}^1 \times \mathbb{I}^2 \tag{6.14}$$

$$(\beta^1_{\tilde{a}^1}, \beta^2_{\tilde{a}^2}) + (\beta^1_{\tilde{a}^1}, \gamma^2_{\tilde{a}^2}) = (\beta^1_{\tilde{a}^1}, \beta^2_{\tilde{a}^2} + \gamma^2_{\tilde{a}^2}) \tag{6.15}$$

$$(\beta^1_{\tilde{a}^1}, \beta^2_{\tilde{a}^2}) + (\gamma^1_{\tilde{a}^1}, \beta^2_{\tilde{a}^2}) = (\beta^1_{\tilde{a}^1} + \gamma^1_{\tilde{a}^1}, \beta^2_{\tilde{a}^2}) \tag{6.16}$$

$$(\beta^1_{\tilde{a}^1}, \beta^2_{\tilde{a}^2}) \bullet (\delta^1_{\tilde{e}^1}, \delta^2_{\tilde{e}^2}) = (\beta^1_{\tilde{a}^1}\delta^1_{\tilde{e}^1}, \beta^2_{\tilde{a}^2}\delta^2_{\tilde{e}^2}) \tag{6.17}$$

$$\kappa(\beta^1_{\tilde{a}^1}, \beta^2_{\tilde{a}^2}) = (\kappa\beta^1_{\tilde{a}^1}, \beta^2_{\tilde{a}^2}) = (\beta^1_{\tilde{a}^1}, \kappa\beta^2_{\tilde{a}^2}) \tag{6.18}$$

Accordingly,

$$(\beta^1_{\tilde{a}^1}, \beta^2_{\tilde{a}^2}) \bullet (\delta^1_{\tilde{e}^1}, \delta^2_{\tilde{e}^2}) = q^{\mathbb{I}^1}_{\tilde{a}^1,\tilde{e}^1}q^{\mathbb{I}^2}_{\tilde{a}^2,\tilde{e}^2}(\delta^1_{\tilde{e}^1}, \delta^2_{\tilde{e}^2}) \bullet (\beta^1_{\tilde{a}^1}, \beta^2_{\tilde{a}^2}) \tag{6.19}$$

$$\begin{aligned}(\beta^1_{\tilde{a}^1}, \beta^2_{\tilde{a}^2}) &\bullet ((\delta^1_{\tilde{e}^1}, \delta^2_{\tilde{e}^2}) \bullet (\theta^1_{\tilde{c}^1}, \theta^2_{\tilde{c}^2})) \\ &= r^{\mathbb{I}^1}_{\tilde{a}^1,\tilde{e}^1,\tilde{c}^1}r^{\mathbb{I}^2}_{\tilde{a}^2,\tilde{e}^2,\tilde{c}^2}((\beta^1_{\tilde{a}^1}, \beta^2_{\tilde{a}^2}) \bullet (\delta^1_{\tilde{e}^1}, \delta^2_{\tilde{e}^2})) \bullet (\theta^1_{\tilde{c}^1}, \theta^2_{\tilde{c}^2})\end{aligned} \tag{6.20}$$

7. BASES AND MULTIPLICATION CONSTANTS FOR GROUP GRADED ALGEBRAS

Every $\mathbb{P}_{\tilde{a}}$, $\tilde{a} \in \mathbb{I}$, of an $(\mathbb{I}; q, r)$-graded algebra over $\mathbb{K}$ is a vector space. So, we can adopt for each $\mathbb{P}_{\tilde{a}} \neq \{0\}$ a maximal set of linearly independent vectors:

$$\{\epsilon_{i_{\tilde{a}}}\} = \{\epsilon_{i_{\tilde{a}}};\ i = 1, \ldots, \dim P_{\tilde{a}}\}, \qquad \text{Hamel basis of } P_{\tilde{a}} \neq \{0\} \tag{7.1}$$

According to the I-graded property (6.7), we have

$$P_{\tilde{a}+\tilde{e}} \ni \epsilon_{i_{\tilde{a}}}\epsilon_{j_{\tilde{e}}} = \sum_{k=1}^{\dim P_{\tilde{a}+\tilde{e}}} \chi_{i_{\tilde{a}}j_{\tilde{e}}}^{k_{\tilde{a}+\tilde{e}}}\epsilon_{k_{\tilde{a}+\tilde{e}}} = \chi_{i_{\tilde{a}}j_{\tilde{e}}}^{k_{\tilde{a}+\tilde{e}}}\epsilon_{k_{\tilde{a}+\tilde{e}}} \tag{7.2}$$

where we adopt as a convention that the *summation runs only over nontilded repeated indices.* The K-valued coefficients $\chi_{i_{\tilde{a}}j_{\tilde{e}}}^{\kappa_{\tilde{a}+\tilde{e}}}$ will determine the product • among arbitrary elements of P in terms of the adopted bases (7.1). The equations (7.2) are called the *multiplication table*, and the K-valued coefficients $\chi_{i_{\tilde{a}}j_{\tilde{e}}}^{\kappa_{\tilde{a}+\tilde{e}}}$ are called the *multiplication constants* of P under the chosen bases. In terms of the multiplication constants, the q-commutativity and r-associativity imply

$$\chi_{i_{\tilde{a}}j_{\tilde{e}}}^{k_{\tilde{a}+\tilde{e}}} = q_{a,e}\chi_{j_{\tilde{e}}i_{\tilde{a}}}^{k_{\tilde{a}+\tilde{e}}} \qquad (q\text{-commutativity}) \tag{7.3}$$

$$\chi_{i_{\tilde{a}}l_{\tilde{e}+\tilde{c}}}^{f_{\tilde{a}+\tilde{e}+\tilde{c}}}\chi_{j_{\tilde{e}}k_{\tilde{c}}}^{l_{\tilde{e}+\tilde{c}}} = r_{\tilde{a},\tilde{e},\tilde{c}}\chi_{i_{\tilde{a}}j_{\tilde{e}}}^{m_{\tilde{a}+\tilde{e}}}\chi_{m_{\tilde{a}+\tilde{e}}k_{\tilde{c}}}^{f_{\tilde{a}+\tilde{e}+\tilde{c}}} \qquad (r\text{-associativity}) \tag{7.4}$$

Observe that the q-commutativity and r-associativity imply the existence of the following constant applications, which we call *q-commutator* and *r-associator*, respectively (They might be useful for defining the (I; q, r)-graded algebra as the quotient of the nonassociative graded algebra by the minimal ideals containing the q-commutators and r-associators):

$$[\![\cdot, \cdot]\!]:\quad P \times P \to \{0\}$$

$$(\beta_{\tilde{a}}, \beta'_{\tilde{e}}) \mapsto [\![\beta_{\tilde{a}}, \beta'_{\tilde{e}}]\!] := \beta_{\tilde{a}}\beta'_{\tilde{e}} - q_{a,e}\beta'_{\tilde{e}}\beta_{\tilde{a}} = 0 \tag{7.5}$$

$$(\cdot, \cdot, \cdot):\quad P \times P \times P \to \{0\}$$

$$(\beta_{\tilde{a}}, \beta'_{\tilde{e}}, \beta''_{\tilde{c}}) \mapsto (\beta_{\tilde{a}}, \beta'_{\tilde{e}}, \beta''_{\tilde{c}})$$

$$:= \beta_{\tilde{a}}(\beta'_{\tilde{e}}\beta''_{\tilde{c}}) - r_{a,\tilde{e},\tilde{c}}(\beta_{\tilde{a}}\beta'_{\tilde{e}})\beta''_{\tilde{c}} = 0 \tag{7.6}$$

8. CONSISTENCY OF (I; q, r)-GRADED ALGEBRAS

We discuss now the consistency of the (I; q, r)-graded parameter algebras over K and provide a consistency property for group gradings. We demonstrate first that every monomial (chain) of parameters built up through binary products • whose factors are clearly delimited by parentheses has a unique result when certain conditions are given.

Definition. An (I q, r)-graded algebra is called *consistent* iff every parameter monomial built up through binary products • is invariant under the q-commutation of factors and r-associative reordering of parentheses.

Proposition 2. The result of a monomial of parameters built up through binary products • whose factors are delimited by parameters *is unique* if the q-commutativity condition (7.3) and the r-associativity condition (7.4) are fulfilled. Hence, every (I; q, r)-graded algebra is consistent.

Proof. We span each parameter $\beta_{\bar{a}}$ involved in the monomial in the terms of the Hamel basis $\{\epsilon_{i\bar{a}}\}$ of $P_{\bar{a}}$: $\beta_{\bar{a}} = c^i \epsilon_{i_{\bar{a}}}$. Then we can use recursively the bilinearity of the • products and the multiplication table (7.2) to transform every product of the form $\beta_{\bar{a}}\beta'_{\bar{e}}$ into

$$\beta_{\bar{a}}\beta'_{\bar{e}} = (c^i \epsilon_{i_{\bar{a}}})(c'^j \epsilon_{j_{\bar{e}}}) = c^i c'^j \chi_{i\bar{a}j\bar{e}}^{k\bar{a}+\bar{e}} \epsilon_{k_{\bar{a}+\bar{e}}} \tag{8.1}$$

Now, each transformation which is done in the calculation using the q-commutativity provides, according to equation (7.3),

$$\beta_{\bar{a}}\beta'_{\bar{e}} = q_{\bar{a},\bar{e}}\beta'_{\bar{e}}\beta_{\bar{a}} = c'^j c^i q_{\bar{a},\bar{e}} \chi_{j\bar{e}i\bar{a}}^{k\bar{a}+\bar{e}} \epsilon_{k_{\bar{a}+\bar{e}}} = c^i c'^j \chi_{i\bar{a}j\bar{e}}^{k\bar{a}+\bar{e}} \epsilon_{k_{\bar{a}+\bar{e}}} \tag{8.2}$$

Hence, the usage of such reordering of factors at any stage of the calculation does not modify the final result in terms of the Hamel basis.

Now, every factor of the form $\beta'_{\bar{a}}(\beta'_{\bar{e}}\beta''_{\bar{c}})$ can be transformed into

$$\beta_{\bar{a}}(\beta'_{\bar{e}}\beta''_{\bar{c}}) = (c^i \epsilon_{i_{\bar{a}}})((c'^j \epsilon_{j_{\bar{e}}})(c''^k \epsilon_{k_{\bar{c}}})) = c^i c'^j c''^k \chi_{i\bar{a}l\bar{e}+\bar{c}}^{f\bar{a}+\bar{e}+\bar{c}} \chi_{j\bar{e}k\bar{c}}^{l\bar{e}+\bar{c}} \epsilon_{f_{\bar{a}+\bar{e}+\bar{c}}} \tag{8.3}$$

The usage of the reordering of parentheses using the r-associativity provides, according to (7.4),

$$\begin{aligned}
\beta_{\bar{a}}(\beta'_{\bar{e}}\beta''_{\bar{c}}) &= r_{\bar{a},\bar{e},\bar{c}}(\beta_{\bar{a}}\beta'_{\bar{e}})\beta''_{\bar{c}} \\
&= c^i c'^j c''^k r_{\bar{a},\bar{e},\bar{c}} \chi_{i\bar{a}j\bar{e}}^{m\bar{a}+\bar{e}} \chi_{m\bar{a}+\bar{e}k\bar{c}}^{f\bar{a}+\bar{e}+\bar{c}} \epsilon_{f_{\bar{a}+\bar{e}+\bar{c}}} \\
&= c^i c'^j c''^k \chi_{i\bar{a}l\bar{e}+\bar{c}}^{f\bar{a}+\bar{e}+\bar{c}} \chi_{j\bar{e}k\bar{c}}^{l\bar{e}+\bar{c}} \epsilon_{f_{\bar{a}+\bar{e}+\bar{c}}}
\end{aligned} \tag{8.4}$$

Hence, the usage of such reordering of parentheses does not modify the final result of the monomial in terms of the Hamel basis.

Since every monomial can be constructed recursively from monomials of lower orders and since the usage of q-commutation factors or r-associative reordering of parentheses has no effect on the partial results, every monomial is determined in an unambiguous way in terms of the Hamel bases. Accordingly, the consistency to all orders is given by the validity of the conditions (7.3) and (7.4). ■

The questions are now:

- Which are sufficient conditions on the q- and r-applications in order to have nontrivial multiplication constants fulfilling (7.3) and (7.4)?
- Which could be a functional relation of the multiplication constants

in terms of the q- and r-applications in order to have conditions (7.3) and (7.4) satisfied?

We consider first the case of an axial (I; q, r)-graded algebra, where the q- and r-applications are given by

$$q_{[\vec{A}],[\vec{E}]} = e^{\vec{\alpha}\cdot(\vec{A}\times\vec{E})} \tag{8.5}$$

$$r_{[\vec{A}],[\vec{E}],[\vec{C}]} = e^{\kappa(\vec{A}\times\vec{C})\cdot\{(\vec{A}+\vec{C})\times\vec{E}\}} \tag{8.6}$$

We first adopt

$$\chi^{[\vec{A}+\vec{E}]}_{[\vec{A}],[\vec{E}]} = (q_{[\vec{A}],[\vec{E}]})^{\text{“}1/2\text{”}} g^{[\vec{A}+\vec{E}]}_{[\vec{E}],[\vec{A}]} = e^{(1/2)\vec{\alpha}\cdot(\vec{A}\times\vec{E})} g^{[\vec{A}+\vec{E}]}_{[\vec{E}],[\vec{A}]} \tag{8.7}$$

Replacing this into (7.3), we conclude

$$g^{[\vec{A}+\vec{E}]}_{[\vec{A}],[\vec{A}]} = g^{[\vec{A}+\vec{E}]}_{[\vec{E}],[\vec{A}]} \tag{8.8}$$

We assume now, using equation (4.14), with $R_{\vec{a},\vec{c}} := (r_{\vec{a},\vec{c}-\vec{a},\vec{c}})^{\text{“}1/4\text{”}}$,

$$g^{[\vec{A}+\vec{E}]}_{[\vec{A}],[\vec{E}]} = R_{[\vec{A}],[\vec{E}]}\rho^{[\vec{A}+\vec{E}]}_{[\vec{A}],[\vec{E}]} = e^{(\kappa/2)\cdot(\vec{A}\times\vec{E})^2}\rho^{[\vec{A}+\vec{E}]}_{[\vec{A}],[\vec{E}]} \tag{8.9}$$

Replacing the χ's in terms of the ρ's in equation (7.4) and using (4.15), we obtain

$$\rho^{[\vec{A}+\vec{E}+\vec{C}]}_{[\vec{A}],[\vec{E}+\vec{C}]}\rho^{[\vec{E}+\vec{C}]}_{[\vec{E}],[\vec{C}]} = \rho^{[\vec{A}+\vec{E}]}_{[\vec{A}],[\vec{E}]}\rho^{[\vec{A}+\vec{E}+\vec{C}]}_{[\vec{A}+\vec{E}],[\vec{C}]} \tag{8.10}$$

A simple choice for the present case would be to take all the ρ constants equal to 1. This choice gives a general solution for axial graded algebras, provided adequate and fixed choices of the phases in (8.7) and (8.9) are adopted.

Analogously, for an iterative (I q, r)-graded algebra, the choice

$$\chi^{l\vec{e}+\vec{c}}_{j\vec{e}k\vec{c}} = (q_{\vec{e},\vec{c}})^{\text{“}1/2\text{”}} R_{\vec{e},\vec{c}}\rho^{l\vec{e}+\vec{c}}_{j\vec{e}k\vec{c}} \tag{8.11}$$

$$\rho^{l\vec{e}+\vec{e}}_{j\vec{e}k\vec{c}} = \rho^{l\vec{e}+\vec{c}}_{k\vec{c}j\vec{e}} \tag{8.12}$$

$$\rho^{f\vec{a}+\vec{e}+\vec{c}}_{i\vec{a}l\vec{e}+\vec{c}}\rho^{l\vec{e}+\vec{c}}_{j\vec{e}k\vec{c}} = \rho^{m\vec{a}+\vec{e}}_{i\vec{a}fj\vec{e}}\rho^{f\vec{a}+\vec{e}+\vec{c}}_{m\vec{a}+\vec{e}k\vec{c}} \tag{8.13}$$

leads to the equations (7.3) and (7.4) for adequate and fixed choices of the phases of the roots in (8.11), (4.14), and (4.15). The ρ-factors are called *branching constants* and the conditions (8.12) and (8.13) [or (8.8) and (8.10)] are called the *branching conditions*. The conditions (4.14)–(4.15) [or (3.28)–(3.29)] of iterative group gradings are used to here to avoid the dependence of the branching constants on the q- and r-factors.

For an (I; q, r)-graded algebra we have $q_{a,\vec{a}} \in \{+1, -1\}$. Since $q_{a,\vec{a}} = -1$ implies nilpotency, we should adopt

$$\rho^{l_{\tilde{e}}+\tilde{c}}_{j\tilde{e}k\tilde{c}} = (1 + \tfrac{1}{2}(q_{\tilde{c},\tilde{c}} - 1)\delta_{jk}\delta_{\tilde{e},\tilde{c}})\,\rho^{l_{\tilde{e}}+\tilde{c}}_{j\tilde{e}k\tilde{c}} \tag{8.14}$$

We have then a recipe for constructing nontrivial iterative graded algebras:

Proposition 3. The iterative (I q, r)-graded algebras over $\mathbb{K}$ are consistent to *every order* if the structure constants are given by (8.11) for fixed choices of phases of the roots that fulfill (4.14)–(4.15), and the branching constants fulfill equations (8.12)–(8.14). This follows from proposition 2 and the construction of structure constants above.

Definition. We call $\{\mathrm{P}; +, \bullet, \cdot\}$ a *quasi-isomorphically* (I q, r)-graded algebra over $\mathbb{K}$ iff:

(i) There are vector spaces $\mathrm{P}_{\tilde{a}}$, $\tilde{a} \in J \subset \mathbb{I}$, such that the whole P is generated through products $\bullet$ of the vectors in the $\mathrm{P}_{\tilde{a}}$; $\tilde{a} \in J \cup \{\tilde{o}\}$ (or in the $\mathrm{P}_{\tilde{a}}$, $\tilde{a} \in J \cup J^{\star} \cup \{\tilde{o}\}$, if P has involution).

(ii) The set J is a minimal basis that generates the whole group $\mathbb{I}$ through group additions.

(iii) Every $\mathrm{P}_{\tilde{a}}$, $\tilde{a} \in \mathbb{I}$, is not trivial; i.e., $\mathrm{P}_{\tilde{a}} \neq \{0\}$, $\tilde{a} \in \mathbb{I}$.

Remark. An abelian-group ring is a trivial quasi-isomorphically graded algebra if its addition is restricted to elements proportional to the same group element and its ring is a field.

Definition We call a group grading ($\mathbb{I}$; q, r) over $\mathbb{K}$ *consistent* iff there exists a quasi-isomorphically ($\mathbb{I}$; q, r)-graded algebra over $\mathbb{K}$.

9. EXAMPLES OF (I; q, r)-GRADED ALGEBRAS OVER C

We list now several ($\mathbb{I}$; q, r)-graded algebras over $\mathbb{C}$ which are of interest in mathematical physics.

Example 3. We consider first the simplest $\mathbb{Z}_2$-graded algebra over $\mathbb{C}$:

$$\text{bosonic basis} \quad \{\mathbf{1}\}; \qquad S_t(\mathbf{1}) = 0 \in \mathbb{Z}_2 \tag{9.1}$$

$$\text{fermionic basis} \quad \{\theta\} \qquad S_t(\theta) = 1 \in \mathbb{Z}_2 \tag{9.2}$$

$$\mathrm{P}'^{\mathbb{Z}_2} = \mathrm{P}'_0 \cup \mathrm{P}'_1 = \{y\mathbf{1}: y \in \mathbb{C}\} \cup \{y\theta: y \in \mathbb{C}\} \tag{9.3}$$

This leads to a faithful, associative, commutative, iterative, quasi-isomorphically ($\mathbb{Z}_2$; $q^{\mathbb{Z}_2}$; $r^{\mathbb{Z}_2}$)-graded algebra over $\mathbb{C}$ when the multiplication table in Table V is adopted.

Proposition 4. ($\mathbb{Z}_2$; $q^{\mathbb{Z}_2}$; $r^{\mathbb{Z}_2}$) is a consistent group grading over $\mathbb{C}$. This follows from $\mathrm{P}'^{\mathbb{Z}_2}$ being quasi-isomorphically graded.

Observe also that due to symmetry of the multiplication table, Table V, $\mathrm{P}'^{\mathbb{Z}_2}$ is commutative.

Table V. Multiplication Table of the Supergraded Algebra P'^{Z_2}

•	1	θ
1	1	θ
θ	θ	0

Example 4. Consider the set of parameters P^{Z_2} with Hamel basis:

$$\text{bosonic basis:} \quad \{\mathbf{1}, \theta_1\theta_2\}; \quad S_t(\mathbf{1}) = S_t(\theta_1\theta_2) = 0 \in Z_2 \tag{9.4}$$

$$\text{fermionic basis:} \quad \{\theta_1, \theta_2\}; \quad S_t(\theta_2) = S_t(\theta_2) = 1 \in Z_2 \tag{9.5}$$

$$P^{Z_2} = P_0 \cup P_1 = \{x\mathbf{1} + y\theta_1\theta_2 \colon x, y \in C\} \cup \{x\theta_1 + y\theta_2 \colon x, y \in C\} \tag{9.6}$$

The algebra P^{Z_2} is a faithful, associative, not commutative, iterative, and quasi-isomorphically $(Z_2; q^{Z_2}; r^{Z_2})$-graded algebra over C when the multiplication table in Table VI is adopted.

It is easy to extend the construction to have a quasi-isomorphically $(Z_2; q^{Z_2}, r^{Z_2})$-graded algebras over C with involution. This is actually the underlying parameter structure of supersymmetry and superspace.

Example 5. We introduce an efficient recipe for determining the structure constants of an axial grading instead of using the general recipe in (8.11)–(8.13). We consider the following set of parameters, whose indices build a minimal set that generates the whole group $Z_{4\Lambda} \times Z_{4\Lambda}$:

$$\{\epsilon_{\vec{e}^0} \equiv \epsilon_{(1,0)}, \epsilon_{\vec{c}^0} \equiv \epsilon_{(0,1)}\} \tag{9.7}$$

We construct a $(Z_{4\Lambda} \times Z_{4\Lambda}; q^{Z^{4\Lambda} \times Z^{4\Lambda}}, r^{Z^{4\Lambda} \times Z^{4\Lambda}})$-graded algebra over C generating the further basis elements in terms of those given in (9.7):

Table VI. Multiplication Table of the Supergraded Algebra P^{Z_2}

•	1	$\theta_1\theta_2$	θ_1	θ_2
1	1	$\theta_1\theta_2$	θ_1	θ_2
$\theta_1\theta_2$	$\theta_1\theta_2$	0	0	0
θ_1	θ_1	0	0	$\theta_1\theta_2$
θ_2	θ_2	0	$-\theta_1\theta_2$	0

$$\epsilon_{(n,m)} := (\epsilon_{(n,0)}\epsilon_{(0,m)}) = ((\epsilon_{(1,0)})^n(\epsilon_{(0,1)})^m) \tag{9.8}$$

We obtain easily the multiplication constants in this case by observing that

$$\begin{aligned}\epsilon_{(n,m)}\epsilon_{(n',m')} &= (\epsilon_{(n,0)}\epsilon_{(0,m)})(\epsilon_{(n',0)}\epsilon_{(0,m')}) \\ &= r^{Z_{4\Lambda}\times Z_{4\Lambda}}_{(n,m),(n',0),(0,m')}(r^{Z_{4\Lambda}\times Z_{4\Lambda}}_{(n,0),(0,m),(n',0)})^{-1} q^{Z_{4\Lambda}\times Z_{4\Lambda}}_{(0,m),(n',0)} r^{Z_{4\Lambda}\times Z_{4\Lambda}}_{(n,0),(n',0),(0,m)} \\ &\quad \times (r^{Z_{4\Lambda}\times Z_{4\Lambda}}_{((n+n') \bmod \Lambda,0),(0,m),(0,m')})^{-1}(\epsilon_{((n+n') \bmod 4\Lambda,0)}\epsilon_{(0,(m+m') \bmod 4\Lambda)}) \\ &= \chi^{((n+n') \bmod 4\Lambda,(m+m') \bmod 4\Lambda)}_{(n,m),(n',m')}\epsilon_{((n+n') \bmod 4\Lambda,(m+m') \bmod 4\Lambda)}\end{aligned} \tag{9.9}$$

Using the definitions (5.4)–(5.5), we obtain an equation for the multiplication constants

$$\begin{aligned}&\chi^{((n+n') \bmod 4\Lambda,(m+m') \bmod 4\Lambda)}_{(n,m),(n',m')} \\ &\quad = \exp\left[\frac{-i\pi}{2\Lambda}\{3nn'mm' + nn'(m^2 + m'^2) + (n^2 + n'^2)mm' + mn'\}\right]\end{aligned} \tag{9.10}$$

from which the q-commutativity and r-associativity conditions (7.3) and (7.4) can be verified. Using this choice for $\Lambda = 1$, we obtain the multiplication table in Table VII. Hence, the set

Table VII. Multiplication Table of $P^{Z_4\times Z_4}$: ($Z_4 \times Z_4$; $q^{Z_4\times Z_4}$, $r^{Z_4\times Z_4}$)-Graded Algebra over C

$\bullet$	$\epsilon_{\tilde{o}}$	$\epsilon_{\tilde{u}1}$	$\epsilon_{\tilde{u}2}$	$\epsilon_{\tilde{u}3}$	$\epsilon_{\tilde{e}0}$	$\epsilon_{\tilde{e}1}$	$\epsilon_{\tilde{e}2}$	$\epsilon_{\tilde{e}3}$	$\epsilon_{\tilde{c}0}$	$\epsilon_{\tilde{c}1}$	$\epsilon_{\tilde{c}2}$	$\epsilon_{\tilde{c}3}$	$\epsilon_{\tilde{s}0}$	$\epsilon_{\tilde{s}1}$	$\epsilon_{\tilde{s}2}$	$\epsilon_{\tilde{s}3}$
$\epsilon_{\tilde{o}}$	$\epsilon_{\tilde{o}}$	$\epsilon_{\tilde{u}1}$	$\epsilon_{\tilde{u}2}$	$\epsilon_{\tilde{u}3}$	$\epsilon_{\tilde{e}0}$	$\epsilon_{\tilde{e}1}$	$\epsilon_{\tilde{e}2}$	$\epsilon_{\tilde{e}3}$	$\epsilon_{\tilde{c}0}$	$\epsilon_{\tilde{c}1}$	$\epsilon_{\tilde{c}2}$	$\epsilon_{\tilde{c}3}$	$\epsilon_{\tilde{s}0}$	$\epsilon_{\tilde{s}1}$	$\epsilon_{\tilde{s}2}$	$\epsilon_{\tilde{s}3}$
$\epsilon_{\tilde{u}1}$	$\epsilon_{\tilde{u}1}$	$\epsilon_{\tilde{o}}$	$\epsilon_{\tilde{u}3}$	$\epsilon_{\tilde{u}2}$	$\epsilon_{\tilde{e}1}$	$\epsilon_{\tilde{e}0}$	$\epsilon_{\tilde{e}3}$	$\epsilon_{\tilde{e}2}$	$\epsilon_{\tilde{c}3}$	$\epsilon_{\tilde{c}2}$	$\epsilon_{\tilde{c}1}$	$\epsilon_{\tilde{c}0}$	$-\epsilon_{\tilde{s}2}$	$-\epsilon_{\tilde{s}3}$	$-\epsilon_{\tilde{s}0}$	$-\epsilon_{\tilde{s}1}$
$\epsilon_{\tilde{u}2}$	$\epsilon_{\tilde{u}2}$	$\epsilon_{\tilde{u}3}$	$\epsilon_{\tilde{o}}$	$\epsilon_{\tilde{u}1}$	$-\epsilon_{\tilde{e}2}$	$-\epsilon_{\tilde{e}3}$	$-\epsilon_{\tilde{e}0}$	$-\epsilon_{\tilde{e}1}$	$\epsilon_{\tilde{c}1}$	$\epsilon_{\tilde{c}0}$	$\epsilon_{\tilde{c}3}$	$\epsilon_{\tilde{c}2}$	$\epsilon_{\tilde{s}3}$	$\epsilon_{\tilde{s}2}$	$\epsilon_{\tilde{s}1}$	$\epsilon_{\tilde{s}0}$
$\epsilon_{\tilde{u}3}$	$\epsilon_{\tilde{u}3}$	$\epsilon_{\tilde{u}2}$	$\epsilon_{\tilde{u}1}$	$\epsilon_{\tilde{o}}$	$-\epsilon_{\tilde{e}3}$	$-\epsilon_{\tilde{e}2}$	$-\epsilon_{\tilde{e}1}$	$-\epsilon_{\tilde{e}0}$	$\epsilon_{\tilde{c}2}$	$\epsilon_{\tilde{c}3}$	$\epsilon_{\tilde{c}0}$	$\epsilon_{\tilde{c}1}$	$-\epsilon_{\tilde{s}1}$	$-\epsilon_{\tilde{s}0}$	$-\epsilon_{\tilde{s}3}$	$-\epsilon_{\tilde{s}2}$
$\epsilon_{\tilde{e}0}$	$\epsilon_{\tilde{e}0}$	$\epsilon_{\tilde{e}1}$	$\epsilon_{\tilde{e}2}$	$\epsilon_{\tilde{e}3}$	$\epsilon_{\tilde{u}1}$	$\epsilon_{\tilde{o}}$	$\epsilon_{\tilde{u}3}$	$\epsilon_{\tilde{u}2}$	$\epsilon_{\tilde{s}1}$	$\epsilon_{\tilde{s}2}$	$-\epsilon_{\tilde{s}0}$	$-\epsilon_{\tilde{s}3}$	$i\epsilon_{\tilde{c}1}$	$-i\epsilon_{\tilde{c}3}$	$-i\epsilon_{\tilde{c}2}$	$i\epsilon_{\tilde{c}0}$
$\epsilon_{\tilde{e}1}$	$\epsilon_{\tilde{e}1}$	$\epsilon_{\tilde{e}0}$	$\epsilon_{\tilde{e}3}$	$\epsilon_{\tilde{e}2}$	$\epsilon_{\tilde{o}}$	$\epsilon_{\tilde{u}1}$	$\epsilon_{\tilde{u}2}$	$\epsilon_{\tilde{u}3}$	$\epsilon_{\tilde{s}3}$	$\epsilon_{\tilde{s}0}$	$-\epsilon_{\tilde{s}2}$	$-\epsilon_{\tilde{s}1}$	$-i\epsilon_{\tilde{c}2}$	$i\epsilon_{\tilde{c}0}$	$i\epsilon_{\tilde{c}1}$	$-i\epsilon_{\tilde{c}3}$
$\epsilon_{\tilde{e}2}$	$\epsilon_{\tilde{e}2}$	$\epsilon_{\tilde{e}3}$	$\epsilon_{\tilde{e}0}$	$\epsilon_{\tilde{e}1}$	$-\epsilon_{\tilde{u}3}$	$-\epsilon_{\tilde{u}2}$	$-\epsilon_{\tilde{u}1}$	$-\epsilon_{\tilde{o}}$	$-\epsilon_{\tilde{s}2}$	$-\epsilon_{\tilde{s}1}$	$-\epsilon_{\tilde{s}3}$	$-\epsilon_{\tilde{o}}$	$i\epsilon_{\tilde{c}0}$	$-i\epsilon_{\tilde{c}2}$	$-i\epsilon_{\tilde{c}3}$	$i\epsilon_{\tilde{c}1}$
$\epsilon_{\tilde{e}3}$	$\epsilon_{\tilde{e}3}$	$\epsilon_{\tilde{e}2}$	$\epsilon_{\tilde{e}1}$	$\epsilon_{\tilde{e}0}$	$-\epsilon_{\tilde{u}2}$	$-\epsilon_{\tilde{u}3}$	$-\epsilon_{\tilde{o}}$	$-\epsilon_{\tilde{u}1}$	$-\epsilon_{\tilde{s}0}$	$-\epsilon_{\tilde{s}3}$	$\epsilon_{\tilde{s}1}$	$\epsilon_{\tilde{s}2}$	$-i\epsilon_{\tilde{c}3}$	$i\epsilon_{\tilde{c}1}$	$i\epsilon_{\tilde{c}0}$	$-i\epsilon_{\tilde{c}2}$
$\epsilon_{\tilde{c}0}$	$\epsilon_{\tilde{c}0}$	$-\epsilon_{\tilde{c}3}$	$\epsilon_{\tilde{c}1}$	$-\epsilon_{\tilde{c}2}$	$-i\epsilon_{\tilde{s}1}$	$i\epsilon_{\tilde{s}3}$	$i\epsilon_{\tilde{s}2}$	$-i\epsilon_{\tilde{s}0}$	$\epsilon_{\tilde{u}2}$	$\epsilon_{\tilde{o}}$	$-\epsilon_{\tilde{u}1}$	$-\epsilon_{\tilde{u}3}$	$-\epsilon_{\tilde{e}1}$	$-\epsilon_{\tilde{e}2}$	$\epsilon_{\tilde{e}0}$	$\epsilon_{\tilde{e}3}$
$\epsilon_{\tilde{c}1}$	$\epsilon_{\tilde{c}1}$	$-\epsilon_{\tilde{c}2}$	$\epsilon_{\tilde{c}0}$	$-\epsilon_{\tilde{c}3}$	$i\epsilon_{\tilde{s}2}$	$-i\epsilon_{\tilde{s}0}$	$-i\epsilon_{\tilde{s}1}$	$i\epsilon_{\tilde{s}3}$	$\epsilon_{\tilde{o}}$	$\epsilon_{\tilde{u}2}$	$-\epsilon_{\tilde{u}3}$	$-\epsilon_{\tilde{u}1}$	$-\epsilon_{\tilde{e}3}$	$-\epsilon_{\tilde{e}0}$	$\epsilon_{\tilde{e}0}$	$\epsilon_{\tilde{e}1}$
$\epsilon_{\tilde{c}2}$	$\epsilon_{\tilde{c}2}$	$-\epsilon_{\tilde{c}1}$	$\epsilon_{\tilde{c}3}$	$-\epsilon_{\tilde{c}0}$	$-i\epsilon_{\tilde{s}0}$	$i\epsilon_{\tilde{s}2}$	$i\epsilon_{\tilde{s}3}$	$-i\epsilon_{\tilde{s}1}$	$\epsilon_{\tilde{u}1}$	$\epsilon_{\tilde{u}3}$	$-\epsilon_{\tilde{u}2}$	$-\epsilon_{\tilde{o}}$	$\epsilon_{\tilde{e}2}$	$\epsilon_{\tilde{e}1}$	$-\epsilon_{\tilde{e}3}$	$-\epsilon_{\tilde{e}0}$
$\epsilon_{\tilde{c}3}$	$\epsilon_{\tilde{c}3}$	$-\epsilon_{\tilde{c}0}$	$\epsilon_{\tilde{c}2}$	$-\epsilon_{\tilde{c}1}$	$i\epsilon_{\tilde{s}3}$	$-i\epsilon_{\tilde{s}1}$	$-i\epsilon_{\tilde{s}0}$	$i\epsilon_{\tilde{s}2}$	$\epsilon_{\tilde{u}3}$	$\epsilon_{\tilde{u}1}$	$-\epsilon_{\tilde{o}}$	$-\epsilon_{\tilde{u}2}$	$\epsilon_{\tilde{e}0}$	$\epsilon_{\tilde{e}3}$	$-\epsilon_{\tilde{e}1}$	$-\epsilon_{\tilde{e}2}$
$\epsilon_{\tilde{s}0}$	$\epsilon_{\tilde{s}0}$	$\epsilon_{\tilde{s}2}$	$-\epsilon_{\tilde{s}3}$	$-\epsilon_{\tilde{s}1}$	$-\epsilon_{\tilde{e}1}$	$-\epsilon_{\tilde{c}2}$	$\epsilon_{\tilde{c}0}$	$\epsilon_{\tilde{c}3}$	$i\epsilon_{\tilde{e}1}$	$-i\epsilon_{\tilde{e}3}$	$-i\epsilon_{\tilde{e}2}$	$i\epsilon_{\tilde{e}0}$	$\epsilon_{\tilde{u}3}$	$-\epsilon_{\tilde{o}}$	$\epsilon_{\tilde{u}2}$	$-\epsilon_{\tilde{u}1}$
$\epsilon_{\tilde{s}1}$	$\epsilon_{\tilde{s}1}$	$\epsilon_{\tilde{s}3}$	$-\epsilon_{\tilde{s}2}$	$-\epsilon_{\tilde{s}0}$	$-\epsilon_{\tilde{c}3}$	$-\epsilon_{\tilde{c}0}$	$\epsilon_{\tilde{c}2}$	$\epsilon_{\tilde{c}1}$	$-i\epsilon_{\tilde{e}2}$	$i\epsilon_{\tilde{e}0}$	$i\epsilon_{\tilde{e}1}$	$-i\epsilon_{\tilde{e}3}$	$-\epsilon_{\tilde{o}}$	$\epsilon_{\tilde{u}3}$	$-\epsilon_{\tilde{u}1}$	$\epsilon_{\tilde{u}2}$
$\epsilon_{\tilde{s}2}$	$\epsilon_{\tilde{s}2}$	$\epsilon_{\tilde{s}0}$	$-\epsilon_{\tilde{s}1}$	$-\epsilon_{\tilde{s}3}$	$\epsilon_{\tilde{c}2}$	$\epsilon_{\tilde{c}1}$	$-\epsilon_{\tilde{c}3}$	$-\epsilon_{\tilde{c}0}$	$i\epsilon_{\tilde{e}0}$	$-i\epsilon_{\tilde{e}2}$	$-i\epsilon_{\tilde{e}3}$	$i\epsilon_{\tilde{e}1}$	$-\epsilon_{\tilde{u}2}$	$\epsilon_{\tilde{u}1}$	$-\epsilon_{\tilde{u}3}$	$\epsilon_{\tilde{o}}$
$\epsilon_{\tilde{s}3}$	$\epsilon_{\tilde{s}3}$	$\epsilon_{\tilde{s}1}$	$-\epsilon_{\tilde{s}0}$	$-\epsilon_{\tilde{s}2}$	$\epsilon_{\tilde{c}0}$	$\epsilon_{\tilde{c}3}$	$-\epsilon_{\tilde{c}1}$	$-\epsilon_{\tilde{c}2}$	$-i\epsilon_{\tilde{e}3}$	$i\epsilon_{\tilde{e}1}$	$i\epsilon_{\tilde{e}0}$	$-i\epsilon_{\tilde{e}2}$	$\epsilon_{\tilde{u}1}$	$-\epsilon_{\tilde{u}2}$	$\epsilon_{\tilde{o}}$	$-\epsilon_{\tilde{u}3}$

$$\mathrm{P}^{Z_{4\Lambda}\times Z_{4\Lambda}} = \{y\epsilon_{\tilde{a}}: y \in \mathrm{C} \text{ and } \tilde{a} \in Z_{4\Lambda} \times Z_{4\Lambda}\} \tag{9.11}$$

with the multiplication constants (9.10) constitutes a faithful, not associative, not commutative, iterative, and quasi-isomophicaly ($Z_{4\Lambda} \times Z_{4\Lambda}$; $q^{Z_{4\Lambda}\times Z_{4\Lambda}}$, $r^{Z_{4\Lambda}\times Z_{4\Lambda}}$)-graded algebra over C.

Proposition 5. ($Z_{4\Lambda} \times Z_{4\Lambda}$; $q^{Z_{4\Lambda}\times Z_{4\Lambda}}$, $r^{Z_{4\Lambda}\times Z_{4\Lambda}}$) is a consistent group grading over C. This follows from the fact that a quasi-isomorphically ($Z_{4\Lambda} \times Z_{4\Lambda}$; $q^{Z_{4\Lambda}\times Z_{4\Lambda}}$, $r^{Z_{4\Lambda}\times Z_{4\Lambda}}$)-graded algebra $\mathrm{P}^{Z_{4\Lambda}\times Z_{4\Lambda}}$ can be constructed.

It is easy to verify that the adopted multiplication constants fulfill (8.11)–(8.14) One realizes as well that the choices of phases in (8.11) for the square and quartic roots are all but trivial.

Example 6. We consider again a set of parameters carrying indices of the group $Z_{4\Lambda} \times Z_{4\Lambda}$:

$$\{\epsilon_{\tilde{e}^0} \equiv \epsilon_{(1,0)}, \epsilon_{\tilde{c}^0} \equiv \epsilon_{0,1}\} \tag{9.12}$$

We construct a ($Z_{4\Lambda} \times Z_{4\Lambda}$; $q^{Z_{4\Lambda}\times Z_{4\Lambda}}$, $r^{(\mathrm{null})}$)-graded algebra over C in an analogous way as we did in (9.8)–(9.10) for Example 5 above. The multiplication constants in this case become

$$\chi_{(n,m),(n',m')}^{((n+n') \bmod 4\Lambda, (m+m') \bmod 4\Lambda)} = e^{(-i\pi/2\Lambda)\{mn,\}} \tag{9.13}$$

from which the q-commutativity and r-associativity conditions (7.3) and (7.4) can be verified. Using this choice of $\Lambda = 1$, we obtain the multiplication table given in Table VIII. Hence, the set

$$\mathrm{P}_{\mathrm{assoc}}^{Z_{4\Lambda}\times Z_{4\Lambda}} = \{y\epsilon_{\tilde{a}}: y \in \mathrm{C} \text{ and } \tilde{a} \in Z_{4\Lambda} \times Z_{4\Lambda}\} \tag{9.14}$$

with the multiplication constants in equations (9.13) constitutes a faithful, associative, not commutative, iterative, and quasi-isomophically ($Z_{4\Lambda} \times Z_{4\Lambda}$; $q^{Z_{4\Lambda}\times Z_{4\Lambda}}$, $r^{(\mathrm{null})}$)-graded algebra over C.

Proposition 6. ($Z_{4\Lambda} \times Z_{4\Lambda}$; $q^{Z_{4\Lambda}\times Z_{4\Lambda}}$, $r^{(\mathrm{null})}$) is a consistent group grading over C. This follows from the fact that a quasi-isomorphically ($Z_{4\Lambda} \times Z_{4\Lambda}$; $q^{Z_{4\Lambda}\times Z_{4\Lambda}}$, $r^{Z_{4\Lambda}\times Z_{4\Lambda}}$)-graded algebra $\mathrm{P}_{\mathrm{assoc}}^{Z_{4\Lambda}\times Z_{4\Lambda}}$ can be constructed.

Example 7. We can consider the direct product of the Z_2-graded algebras and the $Z_4 \times Z_4$-graded algebras. These direct products provide a wide family of parameters adequate for (I; q, r)-graded Lie algebraic extensions of the Poincaré algebra in 3+1 space-time dimensions. According to Wills-Toro (1995), we assign the indices (0, $\tilde{o}$), (o, $\tilde{u}^1$), (0, $\tilde{u}^2$), (0, $\tilde{u}^3$), respectively, to the parameters t, x, y, z of the Minkowski space. The Poincaré transformations use the same set of tilded indices as the spacetime coordinates. The Poincaré generators leave invariant the multiplets, so the horizontal spacings in Tables

Table VIII. Multiplication Table of $P^{Z_4\times Z_4}$: $(Z_4 \times Z_4;\ q^{Z_4\times Z_4},\ r^{(\text{null})})$-Graded Algebra over C.

$\bullet$	$\epsilon_{\tilde{0}}$	$\epsilon_{\tilde{u}^1}$	$\epsilon_{\tilde{u}^2}$	$\epsilon_{\tilde{u}^3}$	$\epsilon_{\tilde{e}^0}$	$\epsilon_{\tilde{e}^1}$	$\epsilon_{\tilde{e}^2}$	$\epsilon_{\tilde{e}^3}$	$\epsilon_{\tilde{c}^0}$	$\epsilon_{\tilde{c}^1}$	$\epsilon_{\tilde{c}^2}$	$\epsilon_{\tilde{c}^3}$	$\epsilon_{\tilde{s}^0}$	$\epsilon_{\tilde{s}^1}$	$\epsilon_{\tilde{s}^2}$	$\epsilon_{\tilde{s}^3}$
$\epsilon_{\tilde{0}}$	$\epsilon_{\tilde{0}}$	$\epsilon_{\tilde{u}^1}$	$\epsilon_{\tilde{u}^2}$	$\epsilon_{\tilde{u}^3}$	$\epsilon_{\tilde{e}^0}$	$\epsilon_{\tilde{e}^1}$	$\epsilon_{\tilde{e}^2}$	$\epsilon_{\tilde{e}^3}$	$\epsilon_{\tilde{c}^0}$	$\epsilon_{\tilde{c}^1}$	$\epsilon_{\tilde{c}^2}$	$\epsilon_{\tilde{c}^3}$	$\epsilon_{\tilde{s}^0}$	$\epsilon_{\tilde{s}^1}$	$\epsilon_{\tilde{s}^2}$	$\epsilon_{\tilde{s}^3}$
$\epsilon_{\tilde{u}^1}$	$\epsilon_{\tilde{u}^1}$	$\epsilon_{\tilde{0}}$	$\epsilon_{\tilde{u}^3}$	$\epsilon_{\tilde{u}^2}$	$\epsilon_{\tilde{e}^1}$	$\epsilon_{\tilde{e}^0}$	$\epsilon_{\tilde{e}^3}$	$\epsilon_{\tilde{e}^2}$	$\epsilon_{\tilde{c}^2}$	$\epsilon_{\tilde{c}^2}$	$\epsilon_{\tilde{c}^1}$	$\epsilon_{\tilde{c}^0}$	$\epsilon_{\tilde{s}^2}$	$\epsilon_{\tilde{s}^3}$	$\epsilon_{\tilde{s}^0}$	$\epsilon_{\tilde{s}^1}$
$\epsilon_{\tilde{u}^2}$	$\epsilon_{\tilde{u}^2}$	$\epsilon_{\tilde{u}^3}$	$\epsilon_{\tilde{0}}$	$\epsilon_{\tilde{u}^1}$	$-\epsilon_{\tilde{e}^2}$	$-\epsilon_{\tilde{e}^3}$	$-\epsilon_{\tilde{e}^0}$	$-\epsilon_{\tilde{e}^1}$	$\epsilon_{\tilde{c}^1}$	$\epsilon_{\tilde{c}^0}$	$\epsilon_{\tilde{c}^3}$	$\epsilon_{\tilde{c}^2}$	$-\epsilon_{\tilde{s}^3}$	$-\epsilon_{\tilde{s}^2}$	$-\epsilon_{\tilde{s}^1}$	$-\epsilon_{\tilde{s}^0}$
$\epsilon_{\tilde{u}^3}$	$\epsilon_{\tilde{u}^3}$	$\epsilon_{\tilde{u}^2}$	$\epsilon_{\tilde{u}^1}$	$\epsilon_{\tilde{0}}$	$-\epsilon_{\tilde{e}^3}$	$-\epsilon_{\tilde{e}^2}$	$-\epsilon_{\tilde{e}^1}$	$-\epsilon_{\tilde{e}^0}$	$\epsilon_{\tilde{c}^2}$	$\epsilon_{\tilde{c}^3}$	$\epsilon_{\tilde{c}^0}$	$\epsilon_{\tilde{c}^1}$	$-\epsilon_{\tilde{s}^1}$	$-\epsilon_{\tilde{s}^0}$	$-\epsilon_{\tilde{s}^3}$	$-\epsilon_{\tilde{s}^2}$
$\epsilon_{\tilde{e}^0}$	$\epsilon_{\tilde{e}^0}$	$\epsilon_{\tilde{e}^1}$	$\epsilon_{\tilde{e}^2}$	$\epsilon_{\tilde{e}^3}$	$\epsilon_{\tilde{u}^1}$	$\epsilon_{\tilde{0}}$	$\epsilon_{\tilde{u}^3}$	$\epsilon_{\tilde{u}^2}$	$\epsilon_{\tilde{s}^1}$	$\epsilon_{\tilde{s}^2}$	$\epsilon_{\tilde{s}^0}$	$\epsilon_{\tilde{s}^3}$	$\epsilon_{\tilde{c}^1}$	$\epsilon_{\tilde{c}^3}$	$\epsilon_{\tilde{c}^2}$	$\epsilon_{\tilde{c}^0}$
$\epsilon_{\tilde{e}^1}$	$\epsilon_{\tilde{e}^1}$	$\epsilon_{\tilde{e}^0}$	$\epsilon_{\tilde{e}^3}$	$\epsilon_{\tilde{e}^2}$	$\epsilon_{\tilde{0}}$	$\epsilon_{\tilde{u}^1}$	$\epsilon_{\tilde{u}^2}$	$\epsilon_{\tilde{u}^3}$	$\epsilon_{\tilde{s}^3}$	$\epsilon_{\tilde{s}^0}$	$\epsilon_{\tilde{s}^2}$	$\epsilon_{\tilde{s}^1}$	$\epsilon_{\tilde{c}^2}$	$\epsilon_{\tilde{c}^0}$	$\epsilon_{\tilde{c}^1}$	$\epsilon_{\tilde{c}^3}$
$\epsilon_{\tilde{e}^2}$	$\epsilon_{\tilde{e}^2}$	$\epsilon_{\tilde{e}^3}$	$\epsilon_{\tilde{e}^0}$	$\epsilon_{\tilde{e}^1}$	$-\epsilon_{\tilde{u}^3}$	$-\epsilon_{\tilde{u}^3}$	$-\epsilon_{\tilde{u}^1}$	$-\epsilon_{\tilde{0}}$	$\epsilon_{\tilde{s}^2}$	$\epsilon_{\tilde{s}^1}$	$\epsilon_{\tilde{s}^3}$	$\epsilon_{\tilde{s}^0}$	$-\epsilon_{\tilde{c}^0}$	$-\epsilon_{\tilde{c}^2}$	$-\epsilon_{\tilde{c}^3}$	$-\epsilon_{\tilde{c}^1}$
$\epsilon_{\tilde{e}^3}$	$\epsilon_{\tilde{e}^3}$	$\epsilon_{\tilde{e}^2}$	$\epsilon_{\tilde{e}^1}$	$\epsilon_{\tilde{e}^0}$	$-\epsilon_{\tilde{u}^2}$	$-\epsilon_{\tilde{u}^3}$	$-\epsilon_{\tilde{0}}$	$-\epsilon_{\tilde{u}^1}$	$\epsilon_{\tilde{s}^0}$	$\epsilon_{\tilde{s}^3}$	$\epsilon_{\tilde{s}^1}$	$\epsilon_{\tilde{s}^2}$	$-\epsilon_{\tilde{c}^3}$	$-\epsilon_{\tilde{c}^1}$	$-\epsilon_{\tilde{c}^0}$	$-\epsilon_{\tilde{c}^2}$
$\epsilon_{\tilde{c}^0}$	$\epsilon_{\tilde{c}^0}$	$-\epsilon_{\tilde{c}^3}$	$\epsilon_{\tilde{c}^1}$	$-\epsilon_{\tilde{c}^2}$	$-i\epsilon_{\tilde{s}^1}$	$i\epsilon_{\tilde{s}^3}$	$-i\epsilon_{\tilde{s}^2}$	$i\epsilon_{\tilde{s}^0}$	$\epsilon_{\tilde{u}^2}$	$\epsilon_{\tilde{0}}$	$-\epsilon_{\tilde{u}^1}$	$-\epsilon_{\tilde{u}^3}$	$i\epsilon_{\tilde{e}^1}$	$-i\epsilon_{\tilde{e}^2}$	$-i\epsilon_{\tilde{e}^0}$	$i\epsilon_{\tilde{e}^3}$
$\epsilon_{\tilde{c}^1}$	$\epsilon_{\tilde{c}^1}$	$-\epsilon_{\tilde{c}^2}$	$\epsilon_{\tilde{c}^0}$	$-\epsilon_{\tilde{c}^3}$	$i\epsilon_{\tilde{s}^2}$	$-\epsilon_{\tilde{s}^0}$	$i\epsilon_{\tilde{s}^1}$	$-i\epsilon_{\tilde{s}^3}$	$\epsilon_{\tilde{0}}$	$\epsilon_{\tilde{u}^2}$	$-\epsilon_{\tilde{u}^3}$	$-\epsilon_{\tilde{u}^1}$	$-i\epsilon_{\tilde{e}^3}$	$i\epsilon_{\tilde{e}^0}$	$i\epsilon_{\tilde{e}^2}$	$-\epsilon_{\tilde{e}^1}$
$\epsilon_{\tilde{c}^2}$	$\epsilon_{\tilde{c}^2}$	$-\epsilon_{\tilde{c}^1}$	$\epsilon_{\tilde{c}^3}$	$-\epsilon_{\tilde{c}^0}$	$i\epsilon_{\tilde{s}^0}$	$-i\epsilon_{\tilde{s}^2}$	$i\epsilon_{\tilde{s}^3}$	$-i\epsilon_{\tilde{s}^1}$	$\epsilon_{\tilde{u}^1}$	$\epsilon_{\tilde{u}^3}$	$-\epsilon_{\tilde{u}^2}$	$-\epsilon_{\tilde{0}}$	$-i\epsilon_{\tilde{e}^2}$	$i\epsilon_{\tilde{e}^1}$	$i\epsilon_{\tilde{e}^3}$	$-i\epsilon_{\tilde{e}^0}$
$\epsilon_{\tilde{c}^3}$	$\epsilon_{\tilde{c}^3}$	$-\epsilon_{\tilde{c}^0}$	$\epsilon_{\tilde{c}^2}$	$-\epsilon_{\tilde{c}^1}$	$-i\epsilon_{\tilde{s}^3}$	$i\epsilon_{\tilde{s}^1}$	$-i\epsilon_{\tilde{s}^0}$	$i\epsilon_{\tilde{s}^2}$	$\epsilon_{\tilde{u}^3}$	$\epsilon_{\tilde{u}^1}$	$-\epsilon_{\tilde{0}}$	$-\epsilon_{\tilde{u}^2}$	$i\epsilon_{\tilde{e}^0}$	$-i\epsilon_{\tilde{e}^3}$	$-i\epsilon_{\tilde{e}^1}$	$i\epsilon_{\tilde{e}^2}$
$\epsilon_{\tilde{s}^0}$	$\epsilon_{\tilde{s}^0}$	$-\epsilon_{\tilde{s}^2}$	$\epsilon_{\tilde{s}^3}$	$-\epsilon_{\tilde{s}^1}$	$i\epsilon_{\tilde{c}^1}$	$-i\epsilon_{\tilde{c}^2}$	$i\epsilon_{\tilde{c}^0}$	$-i\epsilon_{\tilde{c}^3}$	$\epsilon_{\tilde{e}^1}$	$\epsilon_{\tilde{e}^3}$	$-\epsilon_{\tilde{e}^2}$	$-\epsilon_{\tilde{e}^0}$	$-i\epsilon_{\tilde{u}^3}$	$i\epsilon_{\tilde{0}}$	$i\epsilon_{\tilde{u}^2}$	$-i\epsilon_{\tilde{u}^1}$
$\epsilon_{\tilde{s}^1}$	$\epsilon_{\tilde{s}^1}$	$-\epsilon_{\tilde{s}^3}$	$\epsilon_{\tilde{s}^2}$	$-\epsilon_{\tilde{s}^0}$	$-i\epsilon_{\tilde{c}^3}$	$i\epsilon_{\tilde{c}^0}$	$-i\epsilon_{\tilde{c}^2}$	$i\epsilon_{\tilde{c}^1}$	$\epsilon_{\tilde{e}^2}$	$\epsilon_{\tilde{c}^0}$	$-\epsilon_{\tilde{e}^1}$	$-\epsilon_{\tilde{e}^3}$	$i\epsilon_{\tilde{0}}$	$i\epsilon_{\tilde{u}^3}$	$-i\epsilon_{\tilde{u}^1}$	$i\epsilon_{\tilde{u}^2}$
$\epsilon_{\tilde{s}^2}$	$\epsilon_{\tilde{s}^2}$	$-\epsilon_{\tilde{s}^0}$	$\epsilon_{\tilde{s}^1}$	$-\epsilon_{\tilde{s}^3}$	$i\epsilon_{\tilde{c}^2}$	$-i\epsilon_{\tilde{c}^1}$	$i\epsilon_{\tilde{c}^3}$	$-i\epsilon_{\tilde{c}^0}$	$\epsilon_{\tilde{e}^0}$	$\epsilon_{\tilde{e}^2}$	$-\epsilon_{\tilde{e}^3}$	$-\epsilon_{\tilde{e}^1}$	$-i\epsilon_{\tilde{u}^2}$	$i\epsilon_{\tilde{u}^1}$	$i\epsilon_{\tilde{u}^3}$	$-i\epsilon_{\tilde{0}}$
$\epsilon_{\tilde{s}^3}$	$\epsilon_{\tilde{s}^3}$	$-\epsilon_{\tilde{s}^1}$	$\epsilon_{\tilde{s}^0}$	$-\epsilon_{\tilde{s}^2}$	$-i\epsilon_{\tilde{c}^0}$	$i\epsilon_{\tilde{c}^3}$	$-i\epsilon_{\tilde{c}^1}$	$i\epsilon_{\tilde{c}^2}$	$\epsilon_{\tilde{e}^3}$	$\epsilon_{\tilde{e}^1}$	$-\epsilon_{\tilde{e}^0}$	$-\epsilon_{\tilde{e}^2}$	$i\epsilon_{\tilde{u}^1}$	$-i\epsilon_{\tilde{u}^2}$	$-i\epsilon_{\tilde{0}}$	$i\epsilon_{\tilde{u}^3}$

III, IV, VII, and VIII determine four classes of multiplets. The Z_2 factor determine, the commutativity ($q_{\tilde{a},\tilde{a}} = 1$) or anticommutativity ($q_{\tilde{a},\tilde{a}} = -1$) of a parameter with itself, and we call this behavior respectively *self-bosonic* and *self-fermionic*. The parameters associated with self-fermionic multiplets carrying indices of the sets $\{(1, \tilde{u}^\mu): \mu = 0, 1, 2, 3\}$, $\{(1, \tilde{e}^\mu): \mu = 0, 1, 2, 3\}$, $\{(1, \tilde{c}^\mu): \mu = 0, 1, 2, 3\}$ and $\{(1, \tilde{s}^\mu): \mu = 0, 1, 2, 3\}$ are called, respectively, 0-class, 1-class, 2-class, and 3-class of self-fermionic parameters. They resemble several features of quarks. For instance, one can obtain a self-fermionic parameter of the 0-class by a cubic product involving factors of each one of the remaining three classes, in analogy to the composition of a baryon fermion through a triple of quarks of different colors. This is the parameter counterpart of the composition of susy generators through the iterated Lie products of three generators. This is the subject of a forthcoming paper on extended superspace (Wills-Toro, 1997).

Notice that intricate $Z_2 \times (Z_4 \times Z_4)$-graded algebras (with or without involution) might exist that cannot be written as direct products of Z_2- by $Z_4 \times Z_4$-graded algebras.

10. GROUP GRADED OPERATOR ALGEBRAS

To each parameter $\beta_{\tilde{a}} \in P$ of an $(\mathbb{I};\ q,\ r,)$-graded algebra over $\mathbb{K}$ we can associate an operator $L_{\beta_{\tilde{a}}}$, which means "left product by $\beta_{\tilde{a}}$." We can as well define an operator $R_{\beta_{\tilde{a}}}$, which means "right product by $\beta_{\tilde{a}}$":

$$L_{\beta_{\tilde{a}}}(\beta''_{\tilde{u}}) := (\beta_{\tilde{a}}\beta''_{\tilde{u}}) \tag{10.1}$$

$$R_{\beta_{\tilde{a}}}(\beta''_{\tilde{u}}) := (\beta''_{\tilde{u}}\beta_{\tilde{a}}) \tag{10.2}$$

The composition of applications $L_{\beta_{\tilde{a}}}$, and $L_{\beta'_{\tilde{e}}}$, *is not necessarily equivalent to another application of the type* $L_{\gamma_{a+\tilde{e}}}$ *unless the algebra is associative:*

$$L_{\beta_{\tilde{a}}}L_{\beta'_{\tilde{e}}}(\beta''_{\tilde{u}}) = \beta_{\tilde{a}}(\beta'_{\tilde{e}}\beta''_{\tilde{u}}) = r_{\tilde{a},\tilde{e},\tilde{u}}(\beta_{\tilde{a}}\beta'_{\tilde{e}})\beta''_{\tilde{u}} = L_{r_{\tilde{a},\tilde{e},\tilde{u}}\beta_{\tilde{a}}\beta'_{\tilde{e}}}(\beta''_{\tilde{u}}) \tag{10.3}$$

$$L_{\beta_{\tilde{a}}}L_{\beta'_{\tilde{e}}}(\beta'''_{\tilde{c}}) = \beta_{\tilde{a}}(\beta'_{\tilde{e}}\beta'''_{\tilde{c}}) = r_{\tilde{a},\tilde{e},\tilde{c}}(\beta_{\tilde{a}}\beta'_{\tilde{e}})\beta'''_{\tilde{c}} = L_{r_{\tilde{a},\tilde{e},\tilde{c}}\beta_{\tilde{a}}\beta'_{\tilde{e}}}(\beta'''_{\tilde{c}}) \tag{10.4}$$

$$L_{r_{\tilde{a},\tilde{e},\tilde{u}}\beta_{\tilde{a}}\beta'_{\tilde{e}}} \neq L_{r_{\tilde{a},\tilde{e},\tilde{c}}\beta_{\tilde{a}}\beta'_{\tilde{e}}} \quad \text{unless} \quad r_{\tilde{a},\tilde{e},\tilde{u}} = r_{\tilde{a},\tilde{e},\tilde{c}} \tag{10.5}$$

We define a further binary composition "$\oslash$" of operators

$$L_{\beta_{\tilde{a}}} \oslash L_{\beta'_{\tilde{e}}} = L_{\beta_{\tilde{a}}\beta'_{\tilde{e}}} \tag{10.6}$$

which satisfies

$$L_{\beta_{\tilde{a}}}L_{\beta'_{\tilde{e}}} \neq L_{\beta_{\tilde{a}}} \oslash L_{\beta'_{\tilde{e}}} \qquad \text{unless associative parameters} \tag{10.7}$$

$$L_{\beta_{\tilde{a}}} \oslash L_{\beta'_{\tilde{e}}} = q_{a,\tilde{e}}L_{\beta'_{\tilde{e}}} \oslash L_{\beta_{\tilde{a}}} \tag{10.8}$$

$$L_{\beta_{\tilde{a}}} \oslash (L_{\beta'_{\tilde{e}}} \oslash L_{\beta''_{\tilde{c}}}) = r_{\tilde{a},\tilde{e},\tilde{c}}(L_{\beta_{\tilde{a}}} \oslash L_{\beta'_{\tilde{e}}}) \oslash L_{\beta''_{\tilde{c}}} \tag{10.9}$$

In the case in which the parameter algebra is associative, we can interpret the parameter algebra itself as a linear category of multiplications from the left (or from the right). *Remark:* The question of constructing a time-evolution operator (whose composition with the observables provides time-translated observables fulfilling covariant algebraic relations) is guaranteed by the choice of the neutral index for the time parameters and for the time-translation generator.

We will define objects called operators which generalize the above-introduced operators and their compositions.

11. I-GRADED OPERATOR ALGEBRA OVER $\mathbb{K}$

Definition. We call $\{\mathbb{T}; +, \diamond, \cdot\}$ an $\mathbb{I}$-*graded operator algebra over the field* $\mathbb{K}$ iff $\mathbb{T}$ is a set whose elements are called *operators*, $0 \in \mathbb{T}$, $\mathbb{T} \neq \{0\}$, and $\{\mathbb{I}; \wedge\}$ is a group fulfilling the following three axioms:

Axiom 1. There exists an application S_t that assigns an index of $\mathbb{I}$ to each element of $\mathbb{T}\backslash\{0\}$. The index set $\mathbb{I}$ is generated by the set S_t $(\mathbb{T}\backslash\{0\})$ through group operations. The subsets $\mathbb{T}_{\tilde{a}}$ of preimages of each $\tilde{a} \in \mathbb{I}$ with the null element $0 \in \mathbb{T}$ build vector spaces over $\mathbb{K}$:

$$S_t\colon\quad T\backslash\{0\} \to I; \qquad Y_{\tilde{a}} \mapsto S_t(Y_{\tilde{a}}) = \tilde{a} \tag{11.1}$$

$$I \text{ is generated by } S_t(T\backslash\{0\}) \tag{11.2}$$

$$T_{\tilde{a}} = S_t^{-1}(\tilde{a}) \cup \{0\}; \{T_{\tilde{a}}; +, \cdot\} \quad \text{vector space over K} \tag{11.3}$$

$$\{0\} \neq T = \bigcup_{\tilde{a}\in I} T_{\tilde{a}} \tag{11.4}$$

Axiom 2. The product "$\diamond$" is a closed binary I-graded operation in T:

$$\diamond\colon\quad T \times T \to T; \qquad (Y_{\tilde{a}}, Y_{\tilde{e}}) \mapsto Y_{\tilde{a}} \diamond Y_{\tilde{e}} \in T \text{ (closure)} \tag{11.5}$$

$$T_{\tilde{a}} \diamond T_{\tilde{e}} \subset T_{\tilde{a}\wedge\tilde{e}} \qquad \text{(I-grading)} \tag{11.6}$$

Axiom 3. The product "$\diamond$" is bilinear with respect to the operations defined in each vector space $T_{\tilde{a}} \subset T$, i.e., for all $Y_{\tilde{a}}$, $Y'_{\tilde{e}}$, $Y''_{\tilde{e}} \in T$, $y \in K$:

$$Y_{\tilde{a}} \diamond (Y'_{\tilde{e}} + yY''_{\tilde{e}}) = Y_{\tilde{a}} \diamond Y'_{\tilde{e}} + yY_{\tilde{a}} \diamond Y''_{\tilde{e}} \tag{11.7}$$

$$(Y'_{\tilde{e}} + yY''_{\tilde{e}}) \diamond Y_{\tilde{a}} = Y'_{\tilde{e}} \diamond Y_{\tilde{a}} + yY''_{\tilde{e}} \diamond Y_{\tilde{a}} \tag{11.8}$$

We can define a Hamel basis for each nontrivial $T_{\tilde{a}} \subset T$, $\tilde{a} \in I$:

$$\{V_{i\tilde{a}}\} = \{V_{i\tilde{a}}\colon i = 1, \ldots, \dim T_{\tilde{a}}\} \quad \text{Hamel basis for } T_{\tilde{a}} \neq \{0\} \tag{11.9}$$

According to (11.6), we can write

$$V_{i\tilde{a}} \diamond V_{j\tilde{a}} = C^{k\tilde{a}\wedge\tilde{e}}_{i\tilde{a}j\tilde{e}} V_{k\tilde{a}\wedge\tilde{e}} \tag{11.10}$$

where, again, summation over repeated nontilded indices is assumed. The K-valued numbers $C^{k\tilde{a}\wedge\tilde{e}}_{i\tilde{a}j\tilde{e}}$ are called the *structure constants* of the I-graded operator algebra over K. Equations (11.10) constitute the multiplication table of T under the bases (11.9).

Definition. Let $\{T; +, \diamond, \cdot\}$ be an I-graded operator algebra over K. We call T:
associative iff for all $Y_{\tilde{a}}$, $Y'_{\tilde{e}}$, $Y''_{\tilde{c}}$, $\in T$

$$Y_{\tilde{a}} \diamond (Y'_{\tilde{e}} \diamond Y''_{\tilde{c}}) = (Y_{\tilde{a}} \diamond Y'_{\tilde{e}}) \diamond Y''_{\tilde{c}} = Y_{\tilde{a}} \diamond Y'_{\tilde{e}} \diamond Y''_{\tilde{e}} \tag{11.11}$$

with unit $\mathbf{1}_{\tilde{o}}$ iff there exists a unique $\mathbf{1}_{\tilde{o}} \in T_{\tilde{o}}$, $\tilde{o}$ the neutral element of I, such that for all $Y_{\tilde{a}} \in T$

$$Y_{\tilde{a}} \diamond \mathbf{1}_{\tilde{o}} = \mathbf{1}_{\tilde{o}} \diamond Y_{\tilde{a}} = Y_{\tilde{a}} \tag{11.12}$$

q-(anti)symmetric iff additionally (I; q, r) is a faithful iterative group grading over K and for all $Y_{\tilde{a}}$, $Y'_{\tilde{e}} \in T$

$$\Upsilon_{\bar{a}} \diamond \Upsilon'_{\bar{e}} = (-)q_{a,\bar{e}}\Upsilon'_{\bar{e}} \diamond \Upsilon_{\bar{a}} \tag{11.13}$$

q-Jacobi (anti)associative iff (I;q, r) is a faithful associative iterative group grading over K and for all $\Upsilon_{\bar{a}}, \Upsilon'_{\bar{e}}, \Upsilon''_{\bar{c}}, \in \mathbb{T}$

$$\Upsilon_{\bar{a}} \diamond (\Upsilon'_{\bar{e}} \diamond \Upsilon''_{\bar{c}}) = (\Upsilon_{\bar{a}} \diamond \Upsilon'_{\bar{e}}) \diamond \Upsilon''_{\bar{c}} + (-)q_{a,\bar{e}}\Upsilon'_{\bar{e}} \diamond (\Upsilon_{\bar{a}} \diamond \Upsilon''_{\bar{c}}) \tag{11.14}$$

r-associative iff (I; q,r) is a faithful iterative group grading over K and for all $\Upsilon_{\bar{a}}, \Upsilon'_{\bar{e}}, \Upsilon''_{\bar{c}}, \in \mathbb{T}$

$$\Upsilon_{\bar{a}} \diamond (\Upsilon'_{\bar{e}} \diamond \Upsilon''_{\bar{c}}) = r_{\bar{a},\bar{e},\bar{c}}(\Upsilon_{\bar{a}} \diamond \Upsilon'_{\bar{e}}) \diamond \Upsilon'_{\bar{c}} \tag{11.15}$$

(q, r)-Jacobi (anti)associative iff (I; q, r) is a faithful iterative group grading over K and for all $\Upsilon_{\bar{a}}, \Upsilon'_{\bar{e}}, \Upsilon''_{\bar{c}}, \in \mathbb{T}$

$$\Upsilon_{\bar{a}} \diamond (\Upsilon'_{\bar{e}} \diamond \Upsilon_{\bar{c}}) = r_{\bar{a},\bar{e},\bar{c}}(\Upsilon_{\bar{a}} \diamond \Upsilon'_{\bar{e}}) \diamond \Upsilon''_{\bar{c}} + (-)r_{\bar{a},\bar{e},\bar{c}}q_{\bar{a},\bar{e}}r^{-1}_{\bar{e},\bar{a},\bar{c}}\Upsilon'_{\bar{e}} \diamond (\Upsilon_{\bar{a}} \diamond \Upsilon''_{\bar{c}}) \tag{11.16}$$

In terms of the structure constants:

- The associativity implies

$$C^{f\bar{a}\wedge\bar{e}\wedge\bar{c}}_{i\bar{a}m\bar{e}\wedge\bar{c}} C^{m\bar{e}\wedge\bar{c}}_{j\bar{e}k\bar{c}} = C^{l\bar{a}\wedge\bar{e}}_{i\bar{e}j\bar{e}} C^{f\bar{a}\wedge\bar{e}\wedge\bar{c}}_{l\bar{a}\wedge\bar{e}k\bar{c}} \tag{11.17}$$

- The q-(anti)symmetry implies

$$C^{k\bar{a}\wedge\bar{e}}_{i\bar{a}j\bar{e}} = (-)q_{a,\bar{e}}C^{k\bar{e}\wedge\bar{a}}_{j\bar{e}i\bar{a}} \tag{11.18}$$

- The q-Jacobi (anti)associativity implies

$$C^{f\bar{a}\wedge\bar{e}\wedge\bar{c}}_{i\bar{a}l\bar{e}\wedge\bar{c}} C^{l\bar{e}\wedge\bar{c}}_{j\bar{e}k\bar{c}} = C^{m\bar{a}\wedge\bar{e}}_{i\bar{a}j\bar{e}} C^{f\bar{a}\wedge\bar{e}\wedge\bar{c}}_{m\bar{a}\wedge\bar{e}k\bar{c}} + (-)q_{a,\bar{e}} C^{f\bar{a}\wedge\bar{e}\wedge\bar{c}}_{j\bar{e}n\bar{a}\wedge\bar{c}} C^{n\bar{a}\wedge\bar{c}}_{i\bar{a}k\bar{c}} \tag{11.19}$$

- The r-associativity implies

$$C^{f\bar{a}\wedge\bar{e}\wedge\bar{c}}_{i\bar{a}l\bar{e}\wedge\bar{c}} C^{l\bar{e}\wedge\bar{c}}_{j\bar{e}k\bar{c}} = r_{\bar{a},\bar{e},\bar{c}} C^{m\bar{a}\wedge\bar{e}}_{i\bar{a}j\bar{e}} C^{f\bar{a}\wedge\bar{e}\wedge\bar{c}}_{m\bar{a}\wedge\bar{e}k\bar{c}} \tag{11.20}$$

- The (q, r)-Jacobi (anti)associativity implies

$$C^{f\bar{a}\wedge\bar{e}\wedge\bar{c}}_{i\bar{a}l\bar{e}\wedge\bar{c}} C^{l\bar{e}\wedge\bar{c}}_{j\bar{e}k\bar{c}} = r_{\bar{a},\bar{e},\bar{c}} C^{m\bar{a}\wedge\bar{e}}_{i\bar{a}j\bar{e}} C^{f\bar{a}\wedge\bar{e}\wedge\bar{c}}_{m\bar{a}\wedge\bar{e}k\bar{c}} + (-) r_{\bar{a},\bar{e},\bar{c}} q_{\bar{a},\bar{e}} r^{-1}_{\bar{e},\bar{a},\bar{c}} C^{f\bar{a}\wedge\bar{e}\wedge\bar{c}}_{j\bar{e}n\bar{a}\wedge\bar{c}} C^{n\bar{a}\wedge\bar{c}}_{i\bar{a}k\bar{c}} \tag{11.21}$$

Observe that an (I; q, r)-graded algebra over K is just an I-graded operator algebra over K for which (I; q, r) is a group grading over K and whose product is q-symmetric and r-associative.

12. MIXING PARAMETERS AND OPERATORS

We want to introduce a particular requirement on the composition "$\circ$" of $\{\mathbb{T}; +, \circ, \cdot\}$, an I-graded operator algebra over K. We require that the set

$L_{\mathbb{P}}$ of left multiplications by parameters of P, a faithful iterative (I; q, r)-graded parameter algebra over K, is in $\mathbb{T}$:

$$L_{\mathbb{P}} = \{L_{\beta_{\bar{a}}}: \beta_{\bar{a}} \in \mathbb{P}\} \subset \mathbb{T} \tag{12.1}$$

We require as well that the composition "$\circ$" restricted to $L_{\mathbb{P}} \times L_{\mathbb{P}}$ coincides with the composition "$\oslash$" of left products defined in (10.6):

$$\circ|_{L_{\mathbb{P}}^2} = \oslash|_{L_{\mathbb{P}}^2} \tag{12.2}$$

Finally, we require that the composition "$\circ$" between an operator and an element of $L_{\mathbb{P}}$ fulfills q-symmetry:

$$L_{\beta_{\bar{a}}} \circ \Upsilon_{\bar{e}} = q_{a,e}\Upsilon_{\bar{e}} \circ L_{\beta_{\bar{a}}} \qquad \text{for all} \quad L_{\beta_{\bar{a}}}, \Upsilon_{\bar{e}} \in \mathbb{T} \tag{12.3}$$

Definition. We call $L_{\mathbb{P}_{\bar{a}}} \circ \mathbb{T}_{\bar{e}}$ a *linear hull* iff it is the set of all finite linear combinations of elements of the form $L_{\beta_{\bar{a}}} \circ \Upsilon_{\bar{e}} \in \mathbb{T}$:

$$L_{\mathbb{P}_{\bar{a}}} \circ \mathbb{T}_{\bar{e}} = \{L_{\beta^{(1)}_{\bar{a}}} \circ \Upsilon^{(1)}_{\bar{e}} + \cdots + L_{\beta^{(n)}_{\bar{a}}} \circ \Upsilon^{(n)}_{\bar{e}}:$$

$$\beta^{(1)}_{\bar{a}}, \ldots, \beta^{(n)}_{\bar{a}} \in \mathbb{P}_{\bar{a}} \qquad \text{and} \quad \Upsilon^{(1)}_{\bar{e}}, \ldots, \Upsilon^{(n)}_{\bar{e}} \in \mathbb{T}_{\bar{e}}\} \tag{12.4}$$

Obviously, $L_{\mathbb{P}_{\bar{a}}} \circ \mathbb{T}_{\bar{e}} \subset \mathbb{T}_{\bar{a}\wedge\bar{e}}$ and

$$\{L_{\mathbb{P}_{\bar{a}}} \circ \mathbb{T}_{\bar{e}}; +, \cdot\} \quad \text{vector subspace over K of } \mathbb{T}_{\bar{a}\wedge\bar{e}} \tag{12.5}$$

We observe now that for $\bar{o}$, the neutral element of I, the vector space is closed under the composition "$\circ$"; hence

$$\{\mathbb{T}_{\bar{o}}; +, \circ|_{\mathbb{T}_{\bar{o}}^2}, \cdot\} \quad \text{algebra over } \mathbb{K} \tag{12.6}$$

The sets $L_{\mathbb{P}_{-\bar{a}}} \circ \mathbb{T}_{\bar{a}}$, $\bar{a} \in$ I, are vector subspaces of $\mathbb{T}_{\bar{o}}$. Moreover, we can consider the set

$$(L_{\mathbb{P}} \circ \mathbb{T})_{\bar{o}} = \mathbb{K} \bigcup_{\bar{a}\in \mathrm{I}} L_{\mathbb{P}_{-\bar{a}}} \circ \mathbb{T}_{\bar{a}} = \{L_{\beta^{(1)}_{-\bar{a}}} \circ \Upsilon^{(1)}_{\bar{a}} + \cdots + L_{\beta^{(n)}_{-\bar{e}}} \circ \Upsilon^{(n)}_{\bar{e}}:$$

$$\beta^{(1)}_{-\bar{a}}, \ldots, \beta^{(n)}_{-\bar{e}} \in \mathbb{P} \qquad \text{and} \quad \Upsilon^{(1)}_{\bar{a}}, \ldots, \Upsilon^{(n)}_{\bar{e}} \in \mathbb{T}\} \tag{12.7}$$

Obviously, the set $(L_{\mathbb{P}} \circ \mathbb{T})_{\bar{o}} \subset \mathbb{T}_{\bar{o}}$ and constitutes a vector space:

$$\{(L_{\mathbb{P}} \circ \mathbb{T})_{\bar{o}}; +, \cdot\} \quad \text{vector subspace over K of } \mathbb{T}_{\bar{o}} \tag{12.8}$$

We consider now sufficient conditions on a composition "$\diamond$" among operators in order to have $(L_{\mathbb{P}} \circ \mathbb{T})_{\bar{o}}$ an algebra over K. We assume for instance the requirement

$$(L_{\beta_{-a}} \circ \Upsilon_{\bar{a}}) \diamond (L_{\beta'_{\bar{a}}} \circ \Upsilon'_{\bar{e}})$$

$$= r_{-\bar{e},-\bar{a},\bar{a}} r^{-1}_{-\bar{e}-\bar{a},\bar{a},\bar{e}} (L_{\beta'_{-\bar{e}}} \oslash L_{\beta_{-\bar{a}}}) \circ (\Upsilon_{\bar{a}} \diamond \Upsilon'_{\bar{e}}) \in (L_{\mathbb{P}} \circ \mathbb{T})_{\bar{o}} \tag{12.9}$$

It is easy verify that when (12.9) holds and "$\Diamond$" is an internal operation in $\mathbb{T}$, we have

$$\{(L_{\mathrm{P}} \circ \mathrm{T})_{\tilde{\sigma}};\ +,\ \Diamond,\ \cdot\} \quad \text{algebra over } \mathbb{K} \tag{12.10}$$

We study now the constraints of the algebra $(L_{\mathrm{P}} \circ \mathrm{T})_{\tilde{\sigma}}$ in order to arrive at a relation to Lie algebras.

13. GROUP GRADED LIE ALGEBRAIC STRUCTURES

We want to construct now a particular subalgebra of the algebra $\{(L_{\mathrm{P}} \circ \mathrm{T})_{\tilde{\sigma}}; +, \Diamond, \cdot\}$ considered in the previous section. In particular, we address the case in which the composition "$\Diamond$" builds a Lie algebraic substructure that clearly delimits the mixture between elements of L_{P} and a set of operators $\mathrm{L} \subset \mathrm{T}$.

We consider a subset $(L_{\mathrm{P}} \circ \mathrm{L})_{\tilde{\sigma}}$ of the algebra $\{(L_{\mathrm{P}} \circ \mathrm{T})_{\tilde{\sigma}}; +, \Diamond, \cdot\}$ such that

$$\mathrm{L} = \bigcup_{\tilde{a} \in \mathrm{I}} (\mathrm{L}_{\tilde{a}}) \subset \mathrm{T} \tag{13.1}$$

$$\{\mathrm{L}_{\tilde{a}};\ +,\ \cdot\} \quad \text{vector subspace over } \mathbb{K} \text{ of } \mathrm{T}_{\tilde{a}}, \quad \tilde{a} \in \mathrm{I} \tag{13.2}$$

Let $\{\mathcal{A}; +, [\cdot,\cdot], \cdot\}$ be a Lie algebra over $\mathbb{K}$. Then for all $X, Y, Z \in \mathcal{A}$, $\alpha \in \mathbb{K}$,

$$[\mathcal{A};\ +,\ \cdot] \quad \text{vector space over } \mathbb{K} \tag{13.3}$$

$$[\mathcal{A}, \mathcal{A}] \subset \mathcal{A} \qquad \text{(closure)} \tag{13.4}$$

$$[X + \alpha Y, Z] = [X, Z] + \alpha[Y, Z] \qquad \text{(linearity)} \tag{13.5}$$

$$[X, Y] = -[Y, X] \qquad \text{(antisymmetry)} \tag{13.6}$$

$$[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]] \qquad \text{(Jacobi associativity)} \tag{13.7}$$

We ask now for sufficient requirements on the set L and P in order to have $\{L_{\mathrm{P}} \circ \mathrm{L})_{\tilde{\sigma}}; +, [\cdot, \cdot], \cdot\}$ a Lie algebra over $\mathbb{K}$, where

$$[\cdot, \cdot] = \Diamond|_{(L_{\mathrm{P}} \circ \mathrm{L})^2_{\tilde{\sigma}}} \tag{13.8}$$

$$[\![\cdot, \cdot]\!] = \Diamond|_{\mathrm{L}^2} \tag{13.9}$$

Hence, the closure (13.4) follows from (12.9):

$$\begin{aligned} &[L_{\beta_{-\tilde{a}}} \circ \mathrm{Y}_{\tilde{a}}, L_{\beta'_{-\tilde{e}}} \circ \mathrm{Y}'_{\tilde{e}}] \\ &\quad = r_{-\tilde{e},-\tilde{a},\tilde{a}} r^{-1}_{-\tilde{e}-\tilde{a},\tilde{a},\tilde{e}} (L_{\beta'_{-\tilde{e}}\beta_{-\tilde{a}}}) \circ [\![\mathrm{Y}_{\tilde{a}}, \mathrm{Y}'_{\tilde{e}}]\!] \in (L_{\mathrm{P}} \circ \mathrm{L})_{\tilde{\sigma}} \end{aligned} \tag{13.10}$$

The adoption of a commutator of the form (3.27) in (13.10) is allowed by the condition on the q- and r-factors in (3.24), which is satisfied by iterative group gradings. The linearity (13.5) follows from the linearity of the composition "$\Diamond$" of the algebra $\{(L_P \circ T)_{\tilde{\delta}}; +, \Diamond, \cdot\}$ over K. The antisymmetry (13.6) implies

$$[L_{\beta-\tilde{a}} \circ Y_{\tilde{a}}, L_{\beta'-\tilde{e}} \circ Y'_{\tilde{e}}] = -[L_{\beta'-\tilde{e}} \circ Y'_{\tilde{e}}, L_{\beta-\tilde{a}} \circ Y_{\tilde{a}}] \tag{13.11}$$

This condition can be supplied by adopting $[\![\cdot, \cdot]\!]$ to be a q-antisymmetric operation in L:

$$[\![Y_{\tilde{a}}, Y'_{\tilde{e}}]\!] = -q_{\tilde{a},\tilde{e}} [\![Y'_{\tilde{e}}, Y_{\tilde{a}}]\!] \tag{13.12}$$

From the Jacobi associativity requirement (13.7) together with the assumptions (12.2) and (13.10) and the conditions on the q- and r-factors in (3.26), which are satisfied by the faithful iterative group grading (I, q, r) over K, we obtain the (q, r)-Jacobi associativity of $[\![\cdot, \cdot]\!]$:

$$\begin{aligned}[\![Y_{\tilde{a}}, [\![Y'_{\tilde{e}}, Y''_{\tilde{c}}]\!]]\!] &= r_{\tilde{a},\tilde{e},\tilde{c}} [\![[\![Y_{\tilde{a}}, Y'_{\tilde{e}}]\!], Y''_{\tilde{c}}]\!] \\ &\quad + r_{\tilde{a},\tilde{e},\tilde{c}} q_{\tilde{a},\tilde{e}} r^{-1}_{\tilde{e},\tilde{a},\tilde{c}} [\![Y'_{\tilde{e}}, [\![Y_{\tilde{a}}, Y''_{\tilde{c}}]\!]]\!]\end{aligned} \tag{13.13}$$

This construction leads to the definition of the following structure:

Definition. We call $\{L; +, [\![\cdot, \cdot]\!], \cdot\}$ an (I, q, r)-*graded Lie algebra over* K iff $0 \in L \neq \{0\}$, (I; q, r) is a faithful iterative group grading over K, and there is an application S_t: $L\backslash\{0\} \to I$ such that the following five axioms are fulfilled:

Axiom 1. The application S_t assigns an index of I to each element of $L\backslash\{0\}$. $S_t(L\backslash\{0\})$ generates the whole group I. The sets $L_{\tilde{a}}$ of preimages of each $\tilde{a} \in I$ with the null element $0 \in L$ are vector spaces over K:

$$S_t\colon\ L\backslash\{0\} \to I;\quad Q_{\tilde{a}} \mapsto S_t(Q_{\tilde{a}}) = \tilde{a} \tag{13.14}$$

$$\text{I is generated by } S_t(L\backslash\{0\}) \tag{13.15}$$

$$L_{\tilde{a}} = S_t^{-1}(\tilde{a}) \cup \{0\};\quad \{L_{\tilde{a}}; +, \cdot\}\quad \text{vector space over K} \tag{13.16}$$

$$L = \bigcup_{\tilde{a} \in I} (L_{\tilde{a}}) \neq \{0\} \tag{13.17}$$

Axiom 2. The product $[\![\cdot, \cdot]\!]$ is a *closed binary* I-*graded* operation in L:

$$[\![\cdot, \cdot]\!]\colon\ L \times L \to L;\quad (Q_{\tilde{a}}, Q'_{\tilde{e}}) \mapsto [\![Q_{\tilde{a}}, Q'_{\tilde{e}}]\!] \in L \tag{13.18}$$

$$[\![L_{\tilde{a}}, L_{\tilde{e}}]\!] \subset L_{\tilde{a}+\tilde{e}} \tag{13.19}$$

Axiom 3. The product $[\![\cdot, \cdot]\!]$ *bilinear* with respect to the addition operation defined in each vector space $L_{\tilde{a}} \subset L$, i.e., for all $Q_{\tilde{a}}, Q'_{\tilde{e}}, Q''_{\tilde{e}}, \in L$, $y \in K$,

$$[\![Q_{\bar{a}}, Q'_{\bar{e}} + yQ''_{\bar{e}}]\!] = [\![Q_{\bar{a}}, Q'_{\bar{e}}]\!] + y[\![Q_{\bar{a}}, Q''_{\bar{e}}]\!] \tag{13.20}$$

$$[\![Q'_{\bar{e}} + yQ''_{\bar{e}}, Q_{\bar{a}}]\!] = [\![Q'_{\bar{e}}, Q_{\bar{a}}]\!] + y[\![Q''_{\bar{e}}, Q_{\bar{a}}]\!] \tag{13.21}$$

Axiom 4. The product $[\![\cdot, \cdot]\!]$ is *q-antisymmetric*, i.e., for all $Q_{\bar{a}}, Q'_{\bar{e}} \in$ L:

$$[\![Q_{\bar{a}}, Q'_{\bar{e}}]\!] = -q_{\bar{a},\bar{e}}[\![Q'_{\bar{e}}, Q_{\bar{a}}]\!] \tag{13.22}$$

Axiom 5. The product $[\![\cdot, \cdot]\!]$ is *(q, r)-Jacobi associative*, i.e., for all $Q_{\bar{a}}, Q'_{\bar{e}}, Q''_{\bar{c}} \in$ L:

$$[\![Q_{\bar{a}}, [\![Q'_{\bar{e}}, Q''_{\bar{c}}]\!]]\!] = r_{\bar{a},\bar{e},\bar{c}}[\![[\![Q_{\bar{a}}, Q'_{\bar{e}}]\!], Q''_{\bar{c}}]\!] + r_{\bar{a},\bar{e},\bar{c}}q_{\bar{a},\bar{e}}r^{-1}_{\bar{e},\bar{a},\bar{c}}[\![Q'_{\bar{e}}, [\![Q_{\bar{a}}, Q''_{\bar{c}}]\!]]\!] \tag{13.23}$$

Observe that an (I; q, r)-graded Lie algebra over K is just an I-graded operator algebra over K for which (I; q, r) is a faithful iterative group grading over K, and whose product is q-antisymmetric and (q, r)-Jacobi associative.

Proposition 7. If {L; +, $[\![\cdot, \cdot]\!]$, $\cdot$} is an (I; q, r)-graded Lie algebra over K and {P; +, $\bullet$, $\cdot$} is a faithful iterative (I; q, r)-graded parameter algebra over K and (13.10) holds, then $(L_P \circ \text{L})_{\bar{0}}$ is a Lie algebra over K. This follows from the construction above.

The Lie group whose Lie algebra is $(L_P \circ \text{L})_{\bar{0}}$ has elements of the form (2.3) in which we replaced the operator "multiplication from the left by a parameter" by the parameter itself. This structure generalizes the concept of symmetry transformations and invariance and was the aim of this study.

Definition. {L; +, $[\![\cdot, \cdot]\!]$, $\cdot$} is called an (I; q, r)-graded Lie algebra over K *with involution* iff the following extra axiom is fulfilled.

Axiom 6. There exist involutions $(\cdot)^\star$, $(\cdot)^*$, and $\overline{(\cdot)}$ in I, K, and L respectively, such that (I; q, r) is a group grading with involution, and

$$\overline{(yQ_{\bar{a}})} = y^*\overline{Q}_{\bar{a}^\star} \tag{13.24}$$

$$\overline{([\![Q_{\bar{a}}, Q'_{\bar{e}}]\!])} = [\![\overline{Q}'_{\bar{e}^\star}, \overline{Q}_{\bar{a}^\star}]\!] \tag{13.25}$$

Observe that the *q-commutator* defined by

$$[\![Q_{\bar{a}}, Q'_{\bar{e}}]\!] = Q_{\bar{a}} \circ Q'_{\bar{e}} - q_{\bar{a},\bar{e}}Q'_{\bar{e}} \circ Q_{\bar{a}} \tag{13.26}$$

provides a model for the "$[\![\cdot, \cdot]\!]$"-product of the (I; q, r)-graded Lie algebras as long as the "$\circ$"-product is r-associative.

14. CONCLUSIONS

The construction above has settled a structure of a continuous group of transformations which involve noncommutative and nonassociative parameters. It has determined the corresponding graded parameter algebra and graded Lie algebraic structure. An appropriate superspace formalism has been made possible. The novel structures might allow for a better understanding of the Lie groups and their relation thorough graded extensions, and suggest a tool for involving discrete transformations in a Lie algebraic language.

The definition of graded Lie algebras with involution provides a powerful realm for building generalized external symmetries beyond supersymmetry. The presented examples of graded algebras of parameters are the basis for an extended superspace. This line of research might lead, on the one hand, to a better understanding of the relation among the internal and external symmetries of the phenomenological models, and on the other hand, it might lead to models of local external symmetry that offer better understanding of the connection between gravity and quantum physics.

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